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Time-like Dual Curves of Constant Breadth in Dual Lorentzian Space

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Abstract

In this work, some characterizations of closed time-like dual curves and time-like dual curves of constant breadth in dual Lorentzian space are presented.

Keywords: *Dual Lorentzian Space, Time-like Dual Curves, Curves of Constant Breadth.*

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Introduction and Preliminaries

Curves of constant breadth were introduced by Euler (1780). Köse (1984) wrote some geometric properties of plane curves of constant breadth. And, in another work Köse (1986) the author extended these properties to the Euclidean 3-Space E^3. Moreover, Fujivara (1914) obtained a problem to determine whether there exist space curve of constant breadth or not, and he defined 'breadth' for space curves and obtained these curves on a surface of constant breadth. Mağden and Köse (1997), studied this kind curves in four dimensional Euclidean space E^4.

Clifford (1873) introduced dual numbers with the set

$$D = \{\hat{x} = x + \xi x^* : x, x^* \in R\}$$

The symbol ξ designates the dual unit with the property $\xi^2 = 0$ for $\xi \neq 0$. Thereafter, A good amount of research work has been done on dual numbers, dual functions and as well as dual curves (Baky, 2002; Köse et. al., 1988; Sezer et. al. 1990). Then, dual angle is introduced, which is defined as $\hat{\theta} = \theta + \xi\theta^*$, where θ is the projected angle between two spears and θ^* is the shortest distance between them. In recent years, dual numbers have been applied to study the motion of a line in space; they seem even to be most appropriate way for this end and they have triggered use of dual numbers in kinematical problems. There exists a vast literature on the subject, for instance (Velkamp, 1976; Yang, 1963).

The theory of relativity opened a door of use of degenerate submanifolds, and the researchers treated some of classical differential geometry topics, extended to Lorentzian manifolds (Uğurlu and Çalışkan, 1996). In light of the existing literature, we deal with time-like dual curves of constant breadth in dual Lorentzian space.
The set D of dual numbers is a commutative ring with the operations (+) and (.). The set

$$D^3 = D \times D \times D = \{\hat{\varphi} = \varphi + \xi\varphi^* : \varphi, \varphi^* \in E^3\}$$

is a module over the ring D, (Sezer et. al., 1990). Let us denote $\hat{a} = a + \xi a^*$ and $\hat{b} = b + \xi b^*$. The Lorentzian inner product of $\hat{a}$ and $\hat{b}$ is defined by

$$\left\langle \hat{a},\hat{b}\right\rangle = \langle a,b\rangle + \xi\left(\left\langle a^{*},b\right\rangle + \left\langle a,b^{*}\right\rangle\right).$$

We call the dual space D^3 together with Lorentzian inner product as dual Lorentzian space and show by D_1^3. We call the elements of D_1^3 the dual vectors. For $\hat{\varphi} \neq 0$, the norm $\|\hat{\varphi}\|$ of $\hat{\varphi}$ is defined by $\|\hat{\varphi}\| = \sqrt{\left|\langle \hat{\varphi},\hat{\varphi}\rangle\right|}$. A dual vector $\hat{\omega} = \omega + \xi\omega^{*}$ is called dual space-like vector if $\langle \hat{\omega},\hat{\omega}\rangle \rangle 0$ or $\hat{\omega} = 0$, dual time-like vector if $\langle \hat{\omega},\hat{\omega}\rangle \langle 0$ and dual null (light-like) vector if $\langle \hat{\omega},\hat{\omega}\rangle = 0$ for $\hat{\omega} \neq 0$. Therefore, an arbitrary dual curve, which is a differentiable mapping onto D_1^3, can locally be dual space-like, dual time-like or dual null, if its velocity vector is respectively, dual space-like, dual time-like or dual null. Besides, for the dual vectors $\hat{a},\hat{b} \in D_1^3$ Lorentzian vector product of dual vectors is defined by

$$\hat{a} \times \hat{b} = a \times b + \xi\left(a^{*} \times b + a \times b^{*}\right),$$

where $a \times b$ is the classical cross product according to signature $(+,+,-)$, (Uğurlu, Çalışkan, 1996).

Let $\hat{\varphi} : I \subset R \to D_1^3$ be a time-like dual curve with the arc length parameter s. Then the unit tangent vector is defined $\dot{\hat{\varphi}} = \hat{t}$, and the principal normal is $\hat{n} = \dfrac{\dot{\hat{t}}}{\hat{\kappa}}$, where $\hat{\kappa}$ is never pure-dual. The function $\hat{\kappa} = \left\|\dot{\hat{t}}\right\| = \kappa + \xi\kappa^{*}$ is called dual curvature of the dual curve $\hat{\varphi}$. Then the binormal vector of $\hat{\varphi}$ is given by the dual vector $\hat{b} = \hat{t} \times \hat{n}$. Hence, the triple $\left\{\hat{t},\hat{n},\hat{b}\right\}$ is called Frenet trihedra at the point $\hat{\varphi}$,and Frenet formulas may be expressed as (Sezer et. al., 1990)

$$\begin{bmatrix} \dot{\hat{t}} \\ \dot{\hat{n}} \\ \dot{\hat{b}} \end{bmatrix} = \begin{bmatrix} 0 & \hat{\kappa} & 0 \\ \hat{\kappa} & 0 & \hat{\tau} \\ 0 & -\hat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix}, \qquad (1)$$

where $\hat{\tau} = \tau + \xi\tau^{*}$ is the dual torsion of the time-like dual curve $\hat{\varphi}$. Here, we suppose, as the curvature, $\hat{\tau}$ is never pure-dual.

2. Time-like Dual Curves of Constant Breadth in D_1^3

Let $\hat{\varphi} = \hat{\varphi}(s)$ be a simple closed time-like dual curve in the space D_1^3. These curves will be denoted by C. The normal plane at every point P on the curve meets the curve at a single point Q other than P. We call the point Q the opposite point of P. We consider a time-like dual curve in the class Γ as in Fujivara (1916) having parallel tangents $\hat{t}$ and $\hat{t}_\zeta$ in opposite directions at the opposite points $\hat{\varphi}$ and $\hat{\zeta}$ of the curve. A simple closed time-like dual curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to dual Frenet frame by the equation

$$\hat{\zeta} = \hat{\varphi} + \hat{\gamma}\,\hat{t} + \hat{\delta}\,\hat{n} + \hat{\lambda}\,\hat{b}, \tag{2}$$

where $\hat{\gamma}, \hat{\delta}$ and $\hat{\lambda}$ are arbitrary functions of s. Differentiating both sides of (2), we get

$$\frac{d\hat{\zeta}}{ds_\zeta}\frac{ds_\zeta}{ds} = (\frac{d\hat{\gamma}}{ds} + \hat{\delta}\hat{\kappa} + 1)\hat{t} + (\frac{d\hat{\delta}}{ds} + \hat{\gamma}\hat{\kappa} - \hat{\lambda}\hat{\tau})\hat{n} + (\frac{d\hat{\lambda}}{ds} + \hat{\delta}\hat{\tau})\hat{b}. \tag{3}$$

We know that $\hat{t}_\zeta = -\hat{t}$. Considering this, we have the following system of equations

$$\begin{cases} \dfrac{d\hat{\gamma}}{ds} = -\hat{\delta}\hat{\kappa} - 1 - \dfrac{ds_\zeta}{ds} \\ \dfrac{d\hat{\delta}}{ds} = -\hat{\gamma}\hat{\kappa} + \hat{\lambda}\hat{\tau} \\ \dfrac{d\hat{\lambda}}{ds} = -\hat{\delta}\hat{\tau} \end{cases}. \tag{4}$$

If we call $\hat{\phi}$ as the angle between the tangent of the curve C at point $\hat{\varphi}$ with a given fixed direction and consider $\dfrac{d\hat{\phi}}{ds} = \hat{\kappa} = \dfrac{1}{\hat{\rho}}$ and $\dfrac{d\phi}{ds_\zeta} = \hat{\kappa}_\zeta = \dfrac{1}{\hat{\rho}_\zeta}$, we have (4) as follow:

$$\begin{cases} \dfrac{d\hat{\gamma}}{d\hat{\phi}} = -\hat{\delta} - f(\hat{\phi}) \\ \dfrac{d\hat{\delta}}{d\hat{\phi}} = -\hat{\gamma} + \hat{\lambda}\hat{\tau}\hat{\rho} \ , \\ \dfrac{d\hat{\lambda}}{d\hat{\phi}} = -\hat{\delta}\hat{\tau}\hat{\rho} \end{cases} \tag{5}$$

where $f(\hat{\phi}) = \hat{\rho} + \hat{\rho}_{\zeta}$. Using system of ordinary differential equations (5), we have the following dual third order differential equation with respect to $\hat{\gamma}$ as

$$\frac{d}{d\hat{\phi}}\left[\frac{\hat{\kappa}}{\hat{\tau}}\left(\frac{df}{d\hat{\phi}} + \frac{d^2\hat{\gamma}}{d\hat{\phi}^2}\right)\right] + \frac{\hat{\tau}}{\hat{\kappa}}\left(f(\hat{\phi}) + \frac{d\hat{\gamma}}{d\hat{\phi}}\right) - \frac{d\hat{\kappa}}{d\hat{\phi}} = 0. \tag{6}$$

Corollary 1: The obtained dual differential equation of third order (6) is a characterization for the simple closed time-like dual curve $\hat{\zeta}$. By means of solution of it, position vector of a simple closed time-like dual curve can be determined.

Let us investigate solution of equation (6) in a special case. Let κ, κ^* and τ, τ^* be constants. Then the equation (6) has the form

$$\frac{d^3\hat{\gamma}}{d\hat{\phi}^3} + \frac{\hat{\tau}^2}{\hat{\kappa}^2}\frac{d\hat{\gamma}}{d\hat{\phi}} + f(\hat{\phi}) = 0. \tag{7}$$

Solution of equation (7) yields the components

$$\begin{cases} \hat{\gamma} = A\cos\dfrac{\hat{\tau}}{\hat{\kappa}}\hat{\phi} + B\sin\dfrac{\hat{\tau}}{\hat{\kappa}}\hat{\phi} - \dfrac{\hat{\kappa}^2}{\hat{\tau}^2} f(\hat{\phi}).\hat{\phi} \\ \hat{\delta} = \dfrac{\hat{\tau}}{\hat{\kappa}}\left(A\sin\dfrac{\hat{\tau}}{\hat{\kappa}}\hat{\phi} - B\cos\dfrac{\hat{\tau}}{\hat{\kappa}}\hat{\phi}\right) + f(\hat{\phi})\left(\dfrac{\hat{\kappa}^2}{\hat{\tau}^2} - 1\right) \\ \hat{\lambda} = \left(\dfrac{\hat{\kappa}^2 + \hat{\tau}^2}{\hat{\kappa}\hat{\tau}}\right)\left[A\cos\dfrac{\hat{\tau}}{\hat{\kappa}}\hat{\phi} + B\sin\dfrac{\hat{\tau}}{\hat{\kappa}}\hat{\phi}\right] - \dfrac{\hat{\kappa}^3}{\hat{\tau}^3} f(\hat{\phi}).\hat{\phi} \end{cases} . \tag{8}$$

Corollary 2: Position vector of a simple closed time-like curve with constant dual curvature and constant dual torsion can be obtained in terms of $(8)_1$, $(8)_2$ and $(8)_3$.

If the distance between opposite points of $\hat{\varphi}$ and $\hat{\zeta}$ is constant, then, we can write that

$$\left\|\hat{\zeta}-\hat{\varphi}\right\| = -\hat{\gamma}^2 + \hat{\delta}^2 + \hat{\lambda}^2 = \text{constant.} \tag{9}$$

Differentiating with respect to $\hat{\phi}$

$$-\hat{\gamma}\frac{d\hat{\gamma}}{d\hat{\phi}} + \hat{\delta}\frac{d\hat{\delta}}{d\hat{\phi}} + \hat{\lambda}\frac{d\hat{\lambda}}{d\hat{\phi}} = 0. \tag{10}$$

By virtue of (5), the differential equation (10) yields

$$\hat{\gamma}\left(\frac{d\hat{\gamma}}{d\hat{\phi}} + \hat{\delta}\right) = 0. \tag{11}$$

Case I: $\hat{\gamma} = 0$. Then, we have other components

$$\begin{cases} \hat{\delta} = -f(\hat{\phi}) \\ \hat{\lambda} = -\dfrac{\hat{\kappa}}{\hat{\tau}}\left(\dfrac{df}{d\hat{\phi}}\right) \end{cases}. \tag{12}$$

Since the following invariant of the time-like dual curves of constant breadth can be written as

$$\hat{\zeta} = \hat{\varphi} - f(\hat{\phi})\hat{n} - \frac{\hat{\kappa}}{\hat{\tau}}\frac{df}{d\hat{\phi}}\hat{b}. \tag{13}$$

Case II: $\dfrac{d\hat{\gamma}}{d\hat{\phi}} = -\hat{\delta}$. That is, $f(\hat{\phi}) = 0$. We have a relation among radii of curvatures as

$$\hat{\rho} + \hat{\rho}_{\zeta} = 0. \tag{14}$$

Using other components, we easily have a third order variable coefficient differential equation with respect to $\hat{\gamma}$

$$\frac{d}{d\hat{\phi}}\left[\frac{\hat{\kappa}}{\hat{\tau}}\left(\hat{\gamma} - \frac{d^2\hat{\gamma}}{d\hat{\phi}^2}\right)\right] - \frac{\hat{\tau}}{\hat{\kappa}}\frac{d\hat{\gamma}}{d\hat{\phi}} = 0. \tag{15}$$

This equation is characterization for the components. However, the general solution of it has not been found. Due to this, we investigate in a special case.

Let us suppose $\hat{\varphi}$ is a time-like dual curve has constant dual curvatures. In this case, we rewrite (15)

$$\frac{d^3\hat{\gamma}}{d\hat{\phi}^3}+\left(\frac{\hat{\tau}^2}{\hat{\kappa}^2}-1\right)\frac{d\hat{\gamma}}{d\hat{\phi}}=0. \tag{16}$$

The general solution of (16) depends on character of $\left(\frac{\hat{\tau}^2}{\hat{\kappa}^2}-1\right)$. Since, we distinguish following subcases.

Case 2.1: $\kappa=\tau$ and $\kappa^*=\tau^*$. Thus $\frac{d^3\hat{\gamma}}{d\hat{\phi}^3}=0$. By this way we have the components

$$\begin{cases}\hat{\gamma}=\frac{s^2}{2}+cs\\ \hat{\delta}=-s-c\\ \hat{\lambda}=\frac{\hat{\kappa}}{\hat{\tau}}(\frac{s^2}{2}+cs-1)\end{cases}. \tag{17}$$

Case 2.2: $\frac{\kappa}{\kappa^*}=\frac{\tau}{\tau^*}$ and $\tau\rangle\kappa$. Then, by means of solution of (16), we have the components

$$\begin{cases}\hat{\gamma}=A\cos\left(\sqrt{\frac{\hat{\tau}^2}{\hat{\kappa}^2}-1}\right)\hat{\phi}+B\sin\left(\sqrt{\frac{\hat{\tau}^2}{\hat{\kappa}^2}-1}\right)\hat{\phi}\\ \hat{\delta}=\left(\sqrt{\frac{\hat{\tau}^2}{\hat{\kappa}^2}-1}\right)\left[A\sin\left(\sqrt{\frac{\hat{\tau}^2}{\hat{\kappa}^2}-1}\right)\hat{\phi}-B\cos\left(\sqrt{\frac{\hat{\tau}^2}{\hat{\kappa}^2}-1}\right)\hat{\phi}\right]\\ \hat{\lambda}=\left(\frac{\hat{\kappa}}{\hat{\tau}}\right)\left(\frac{\hat{\tau}^2}{\hat{\kappa}^2}-1\right)\left[A\cos\left(\sqrt{\frac{\hat{\tau}^2}{\hat{\kappa}^2}-1}\right)\hat{\phi}+B\sin\left(\sqrt{\frac{\hat{\tau}^2}{\hat{\kappa}^2}-1}\right)\hat{\phi}\right]\end{cases}. \tag{18}$$

Case 2.3: $\frac{\kappa}{\kappa^*}=\frac{\tau}{\tau^*}$ and $\tau\langle\kappa$. Consequently, using hyperbolic functions we arrive

$$\begin{cases} \hat{\gamma} = K\cosh\left(\sqrt{1-\dfrac{\hat{\tau}^2}{\hat{\kappa}^2}}\right)\hat{\phi} + L\sinh\left(\sqrt{1-\dfrac{\hat{\tau}^2}{\hat{\kappa}^2}}\right)\hat{\phi} \\ \hat{\delta} = -\left(\sqrt{1-\dfrac{\hat{\tau}^2}{\hat{\kappa}^2}}\right)\left[K\sinh\left(\sqrt{1-\dfrac{\hat{\tau}^2}{\hat{\kappa}^2}}\right)\hat{\phi} + L\cosh\left(\sqrt{1-\dfrac{\hat{\tau}^2}{\hat{\kappa}^2}}\right)\hat{\phi}\right]. \\ \hat{\lambda} = \dfrac{\hat{\kappa}}{\hat{\tau}}\left(\dfrac{\hat{\tau}^2}{\hat{\kappa}^2}-1\right)\left[K\cosh\left(\sqrt{1-\dfrac{\hat{\tau}^2}{\hat{\kappa}^2}}\right)\hat{\phi} + L\sinh\left(\sqrt{1-\dfrac{\hat{\tau}^2}{\hat{\kappa}^2}}\right)\hat{\phi}\right] \end{cases} \quad (19)$$

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