Pauli Matrixes and Generalized Rotations

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Abstract

In the article, using spinor representation of orthogonal transformations, the equations, for generalized rotations are received. The rotations are defined as set of all possible rotations, both with zero, and non-zero centers which carry out transformations of initial 3-dimensional point into a final. The expressions between second order complex unitary transformations matrixes and real orthogonal matrixes of rotations in L^3 are received, that allows easily calculating of corresponding Euler's angles.

Keywords: *Spinors, Generalized Rotations, Hermitian Transformations, Orthogonal Transformations.*

Statement of the Problem

Methods of representation of three-dimensional rotations used in solving various engineering problems are usually confined to the description of individual concrete rotations centered at the origin (zero center). Among these methods is in particular the well known method of orthogonal real matrices whose elements are functions of Euler angles (Gelfand, Minlos and Shapiro, 1958), (Fu, Gonzales and Lee, 1987). At the same time it should be said that the problem of describing so-called generalized rotations (Gelfand, Minlos and Shapiro, 1958) evokes a much greater interest both from the theoretical standpoint and from the standpoint of applications (in the first place we mean an application in robotics and in particular in the planning of trajectories in the case of obstacles). Under generalized rotations we mean the set of all possible rotations with both zero and nonzero centers which transform the initial three-dimensional point to the finite one. The basic problem arising in this context can be formulated as follows: Given two three-dimensional points $x(x^1,x^2,x^3)$ and $y(y^1,y^2,y^3)$, it is required to define the set of all possible transformations and centers of rotations which bring about the transformation of the point x to the point y. It is obvious that this problem can be easily extended to the case

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where instead of two points we consider two finite sets of points $\{x_i\ (x_i^1, x_i^2, x_i^3\}$ and $\{y_i\ (y_i^1, y_i^2, y_i^3\}$ i=1,2,…m, which corresponds to the case of rotations of a solid.

Equations for Generalized Rotations

Let L^3 be a linear Euclidean space with orthonormalized basis vectors e_1, e_2, e_3. To each vector $x=x^1e_1+ x^2e_2+x^3e_3$ of the space L^3 we assign a traceless Hermitian matrix

$$X = \begin{vmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{vmatrix},$$

whose elements are the so-called spinor components of the vector x (Kostrikin and Manin, 1980). When we pass from the usual Euclidean components of the vector x to the spinor ones, we thereby identify the vector x with Hermitian functionals on the two-dimensional linear space C^2 over the field of complex numbers C (Postnikov, 1982). Denote by $L(C^2)$ the set of all Hermitian functionals on C^2 which is a linear three-dimensional space over the field of real numbers provided that Pauli matrices are taken as basis elements. Then for each matrix of form (1) we have the decomposition

$$X= x^1\sigma_1+ x^2\sigma_2+x^3\sigma_3,$$

where

$$\sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

are Pauli matrices.

From decomposition (2) it follows that the set $L(C^2)$ is a linear three-dimensional space over the field of real numbers and thus it can be identified with L^3. Note that to each basis vector of the two-dimensional space C^2 we can assign the basis vectors $\sigma_1,\sigma_2,\sigma_3$ of the space $L(C^2)$ (and also the orthonormalized basis vector e_1,e_2,e_3 due to the identification of L^3 and $L(C^2)$): each of the matrices σ_i is represented as some linear combination of tensor products of basis vectors of the space C^2 (Kostrikin, Manin, 1980). The foregoing reasoning implies that for any matrix $C \in C^2$, which is a matrix of transformation between two basis vectors of the space C^2, there also exists a transformation matrix of the corresponding orthonormalized basis vectors in the space L^3.

Proposition_1. The matrix of transformation of the basis vectors in C^2 is unitary.
Proof. If on the space C^2 we consider Hermitian functionals of the form

$$X = \begin{vmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^o - x^3 \end{vmatrix},$$

then they will correspond to the four-dimensional vectors of a pseudo-Euclidean space with signature (1,3) and with basis vectors

$$\sigma_0 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}; \sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}.$$
(3)

Now, transformations of the basis vectors of the two-dimensional space C^2 lead to transformations of the basis vectors (3), while the transformation matrices remain the same as in the case of functionals of form (1). The orthogonal complement $\perp\sigma_0$ of the first basis vector σ_0 is an anti-Euclidean space (because of the pseudo-Euclidean property of the space defined by vectors (3)) and, after changing the signs of the scalar products, a three-dimensional Euclidean space that coincides with $L(C^2)$. The restriction of the action of matrices of basis vector transformation in C^2 to the subspace $\perp\sigma_0$ means that these matrices satisfy the condition $\overline{C}^T \sigma_0 C = \sigma_0$, i.e. $C^{-1} = \overline{C}^T$, Q.E.D.

The problem posed in Subsection 1 can be now reformulated in terms of the spinor space C^2: Given two traceless matrices of Hermitian functionals

$$X = \begin{vmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{vmatrix} \text{ and и } Y = \begin{vmatrix} y^3 & y^1 - iy^2 \\ y^1 + iy^2 & -y^3 \end{vmatrix},$$

it is required to define:

1) a set of unitary matrices $C = \begin{vmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{vmatrix}$ which satisfy the equality $Y = \overline{C}^T XC$;
2) one-dimensional subspaces which are invariant with respect to transformations represented by matrices C (i.e. a set of respective rotation centers).

Note that since the transformation C is unitary, the vector norms defined by the determinants of matrices of the Hermitian functionals X and Y coincide and therefore (4) defines rotation. From equality (4) we can obtain the following system of linear homogeneous equations with respect to the unknown variables α and β:

$$\begin{aligned} x_3\alpha + \gamma\beta &= y_3\alpha - \overline{\delta\beta} \\ \bar{\gamma}\alpha - x_3\beta &= y_3\beta + \bar{\delta}\bar{\alpha} \end{aligned},$$

where $\gamma = x_1 + ix_2$ and $\delta = y_1 + iy_2$.

For arbitrary α, a solution of (5) is given by

$$\beta = \frac{\bar{\gamma}\alpha - \bar{\delta}\bar{\alpha}}{x_3 + y_3} .$$

From (6) we have

$$\operatorname{Re}\beta = \beta_1 = \frac{\alpha_1(x_1 - y_1) + \alpha_2(x_2 + y_2)}{x_3 + y_3}$$ and

$$\operatorname{Im}\beta = \beta_2 = \frac{\alpha_2(x_1 + y_1) - \alpha_1(x_2 - y_2)}{x_3 + y_3} .$$

Using the unitarity of the matrix C ($\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$), we can define either $\alpha_1 = \operatorname{Re}\alpha$ or $\alpha_2 = \operatorname{Im}\alpha$. Note that one of these parameters remains arbitrary. Thus (6) defines rotation for $\alpha \neq 0$ and $x_3 + y_3 \neq 0$.

The invariance of the rotation center $z(z_1, z_2, z_3)$ with respect to the transformation C is written as a condition

$\overline{C}^T ZC = Z$,

whence we obtain

$$\begin{aligned} z_3\alpha + \mu\beta &= z_3\alpha - \overline{\mu}\overline{\beta} \\ \overline{\mu}\alpha - z_3\beta &= z_3\beta + \overline{\mu}\overline{\alpha} \end{aligned},$$

where $\mu = z_1 + iz_2$.

The latter formula leads to the system

$$\begin{aligned} \beta_1 z_1 - \beta_2 z_2 &= 0; \\ \alpha_2 z_2 - \beta_1 z_3 &= 0; \\ \alpha_1 z_1 - \beta_2 z_3 &= 0. \end{aligned}$$

It is not difficult to verify that the determinant of this system considered for the unknown values z_1, z_2 and z_3 is identically zero and therefore for given α_1, α_2, β_1 and β_2 ($\alpha \neq 0$ and $x_3 + y_3 \neq 0$) there always exist nontrivial solutions written in the form

$$z_1 = \frac{\beta_2}{\alpha_2} z_3; \quad z_2 = \frac{\beta_1}{\alpha_2} z_3,$$

where z_3 is arbitrary.

Equations (7) define the one-parametric set of transformations C_t due to which (x^1, x^2, x^3) changes to (y^1, y^2, y^3) by means of rotation. If we choose α_1 as a parameter, then to its each fixed value defining the unique transformation $C_{t=\alpha_1}$ we can assign the set of rotation centers

$$\alpha_2 z_1 + \alpha_2 z_2 - (\beta_1 + \beta_2) z_3 = 0$$

whose equation is readily obtained from (8).

Thus, (7) together with the normalization condition define a generalized rotation transforming (x^1,x^2,x^3) to (y^1,y^2,y^3) with respect to the set of centers which is defined by (9).

Relations Between Transformations in C^2 and L^3

We can establish the correspondence between the elements of the transformation matrix $C = \begin{vmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{vmatrix}$ in C^2 and the elements of the orthogonal real matrix of rotation A in L^3.

The matrix A is, by definition, the matrix of transformation between two orthonormalized basis vectors of the space L^3 and its rows are decompositions of the new basis vectors in terms of the initial basis vectors. Hence due to the identification of the spaces $L(C^2)$ and L^3 we have

$$\overline{C}^T \sigma_i C = a_i^{i'} \sigma_{i'} \quad (i, i' = 1,2,3),$$

where σ_i are the Pauli matrices corresponding to the initial basis, $\sigma_{i'}$ are the Pauli matrices of the new basis, and $\alpha_i^{i'}$ are the elements of the matrix A^{-1}. Formula (10) can be written explicitly in the form of three matrix equalities

$$a_1^1\begin{vmatrix}0&1\\1&0\end{vmatrix}+a_2^1\begin{vmatrix}0&-i\\i&0\end{vmatrix}+a_3^1\begin{vmatrix}1&0\\0&-1\end{vmatrix}=\begin{vmatrix}\overline{\alpha}&-\beta\\\overline{\beta}&\alpha\end{vmatrix}*\begin{vmatrix}0&1\\1&0\end{vmatrix}*\begin{vmatrix}\alpha&\beta\\-\overline{\beta}&\overline{\alpha}\end{vmatrix},$$

$$a_1^2\begin{vmatrix}0&1\\1&0\end{vmatrix}+a_2^2\begin{vmatrix}0&-i\\i&0\end{vmatrix}+a_3^2\begin{vmatrix}1&0\\0&-1\end{vmatrix}=\begin{vmatrix}\overline{\alpha}&-\beta\\\overline{\beta}&\alpha\end{vmatrix}*\begin{vmatrix}0&-i\\i&0\end{vmatrix}*\begin{vmatrix}\alpha&\beta\\-\overline{\beta}&\overline{\alpha}\end{vmatrix}.$$

$$a_1^3\begin{vmatrix}0&1\\1&0\end{vmatrix}+a_2^3\begin{vmatrix}0&-i\\i&0\end{vmatrix}+a_3^3\begin{vmatrix}1&0\\0&-1\end{vmatrix}=\begin{vmatrix}\overline{\alpha}&-\beta\\\overline{\beta}&\alpha\end{vmatrix}*\begin{vmatrix}1&0\\0&-1\end{vmatrix}*\begin{vmatrix}\alpha&\beta\\-\overline{\beta}&\overline{\alpha}\end{vmatrix}$$

which readily yield the following expressions for calculating the elements of the matrix A by the elements of the matrix C:

$$a_1^1=(\alpha_1^2-\alpha_2^2)-(\beta_1^2-\beta_2^2); \qquad a_2^1=2(\alpha_1\alpha_2+\beta_1\beta_2);$$
$$a_3^1=2(\alpha_2\beta_2-\alpha_1\beta_1);$$
$$a_1^2=2(\beta_1\beta_2-\alpha_1\alpha_2); \qquad a_2^2=(\alpha_1^2-\alpha_2^2)+(\beta_1^2-\beta_2^2);$$
$$a_3^2=2(\alpha_1\beta_2+\alpha_2\beta_1); \quad (11)$$
$$a_1^3=2(\alpha_1\beta_1+\alpha_2\beta_2); \qquad a_2^3=2(\alpha_2\beta_1-\alpha_1\beta_2);$$
$$a_3^3=(\alpha_1^2+\alpha_2^2)-(\beta_1^2+\beta_2^2).$$

Expressions (11) enable us to calculate the elements of the matrix A through the given coordinates of three points (initial, terminal and the center) which define rotation.

On the other hand, taking into account that the matrix A can be written in the form (Gelfand, I.M., Minlos, R.A. and Shapiro, Z.I., 1958)

$$A=\begin{Vmatrix}\cos\varphi\cos\psi-\cos\theta\sin\varphi\sin\psi & -\cos\varphi\sin\psi-\cos\theta\sin\varphi\sin\psi & \sin\varphi\sin\theta\\ \sin\varphi\cos\psi+\cos\theta\cos\varphi\sin\psi & -\sin\varphi\sin\psi+\cos\theta\cos\varphi\cos\psi & -\cos\varphi\sin\theta\\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta\end{Vmatrix}, \quad (12)$$

where $-\pi<\varphi\le\pi$, $0\le\theta\le\pi$ и $-\pi<\psi\le\pi$ are Euler angles, it easily follows that expressions (11) enable us to define Euler angles as well.

A Numerical Example

Given two arbitrary vectors of equal length $x(100, -30, 10)$ and $y(-12, 2, 104.73)$ and with zero center of rotation, we calculate, by (7), $\beta = -6.867+4.765i$ for arbitrary $\alpha = -5+8i$. Hence, after normalization, we obtain the transformation matrix

$$C = \begin{vmatrix} -0.397+0.635i & -0.545+0.378i \\ 0.545+0.378i & -0.397-0.635i \end{vmatrix}.$$

Representing the vectors x and y by the spinor matrices

$$X = \begin{vmatrix} 10 & 100+30i \\ 100-30i & -10 \end{vmatrix} \text{ and } Y = \begin{vmatrix} 104.173 & -12-2i \\ -12+2i & -104.173 \end{vmatrix},$$

we verify the validity of equality (4):

$$\begin{vmatrix} -0.397+0.653i & -0.545+0.378i \\ 0.545+0.378i & -0.397-0.653i \end{vmatrix} * \begin{vmatrix} 10 & 100+30i \\ 100-30i & 10 \end{vmatrix} * \begin{vmatrix} -0.397-0.653i & 0.545-0.378i \\ -0.545-0.378i & -0.397+0.653i \end{vmatrix}$$

$$= \begin{vmatrix} 104.173 & -12-2i \\ -12+2i & -104.173 \end{vmatrix},$$

which means that the transformation matrix C defined in this manner actually represents the sought for rotation. Using formulas (11), we calculate the matrix

$$A = \begin{vmatrix} -0.399 & -0.916 & 0.048 \\ 0.092 & -0.092 & -0.992 \\ 0.912 & -0.392 & 0.121 \end{vmatrix}.$$

It is not difficult to verify that the determinant of the matrix A is equal to unit and that $A^{-1} = A^{T}$, i.e. A is indeed an orthogonal matrix. Further, we verify $A*x = y$:

$$\begin{vmatrix} -0.399 & -0.916 & 0.048 \\ 0.092 & -0.092 & -0.992 \\ 0.912 & -0.392 & 0.121 \end{vmatrix} * \begin{vmatrix} 100 \\ -30 \\ 10 \end{vmatrix} = \begin{vmatrix} -12 \\ 2 \\ 104.73 \end{vmatrix}.$$

Thus the matrix A represents the same rotation in the space L^3 as the complex matrix C in C^2.

Now let us take a nonzero center of rotation lying in plane (9). Assuming that the coordinate z_3 has an arbitrary value $z_3 = 10$, we calculate, by formulas (11), the other two coordinates $z_2 = -8.583$ and $z_1 = 5.956$. We immediately see that

$$\begin{vmatrix} -0.399 & -0.916 & 0.048 \\ 0.092 & -0.092 & -0.992 \\ 0.912 & -0.392 & 0.121 \end{vmatrix} * \begin{vmatrix} 5.956 \\ -8.583 \\ 10 \end{vmatrix} = \begin{vmatrix} 5.956 \\ -8.583 \\ 10 \end{vmatrix},$$

i.e. the vector z is the eigenvector corresponding to eigenvalue 1.

It is likewise easy to establish that the matrix A is the matrix of rotation of the initial points x to the point y with respect to the new center shifted by the vector z with respect to the zero center. To this end, we calculate the vectors $x - z = (94.044, -21.417, 0)$ and $y - z = (-17.956, 10.583, 94.173)$ and the product

$$\begin{vmatrix} -0.399 & -0.916 & 0.048 \\ 0.092 & -0.092 & -0.992 \\ 0.912 & -0.392 & 0.121 \end{vmatrix} * \begin{vmatrix} 94.044 \\ 21.417 \\ 0 \end{vmatrix} = \begin{vmatrix} -17.956 \\ 10.583 \\ 94.173 \end{vmatrix}.$$

Moreover, it is obvious that knowing the numerical values of the elements of the matrix A and taking into account (12), we can easily calculate the corresponding Euler angles, which can be used in future developments of effective methods for optimal control of moving components of various robots.

Conclusion

Equations for generalized rotations are obtained by means of Pauli matrices. Rotations are defined as the set of all possible rotations with both zero and nonzero centers that transform the initial three-dimensional point to the terminal point. Relations between second order complex unitary transformation matrices written in Pauli basis and real orthogonal matrices are established, which enable us to easily calculate the corresponding Euler angles.

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