IBSU Scientific Journal 2009, 3(2), 137-148

A Variational Approach in Drivelines Terminal Control

Cumhur AKSU

Abstract

It is shown that on the base of Driveline acceleration measurement it is possible to define a variational problem, solution of which permits to synthesize general Control function of Drivelines control process. The last function covers various particular control problems. It is shown, that these problems can be naturally defined by choosing of corresponding boundary conditions. As an example of such particular cases a problem of Acceleration is solved.

Cumhur Aksu *is an assistant professor and PhD candidate at Faculty of Computer Sciences and Engineering of International Black Sea University, Tbilisi, Georgia*

Introduction

Problems connected to the elaboration of methods and algorithms for **Control of Drivelines** belong to the class of one of the most difficult mathematical and engineering problems and at the same time they are widely investigated problems due their practical importance (Mchedlishvili, 2008, Petrov, 1971). Despite of it, majority of existing approaches of solution of the mentioned problems can be characterized as complicate, from their mathematical foundations, and that is why difficult to realize. In the first place, they include such methods as the principle of maximum, dynamic programming, the momentum method and others directly connected with the classical methods of variational calculus. These methods are rather difficult for application, since the eventual control algorithms obtained with their aid are actually of programming character, i.e. explicitly depending on time. Therefore it is impossible to carry out the current correction of a phase trajectory, though such a correction is absolutely necessary because a moving object is influenced by perturbing environmental factors (both systematic and random).

Definition of the problem

Below we will consider only one-dimensional problems of adaptive control of **Drivelines.** In the Fig.1 the simplest principal scheme of such a Driveline is shown. The load of the motor is reduced to simplify discussion of general principles and approaches given below. Later, having in view application problems of certain drivelines units, we shall analyze structured (not reduced) loads.

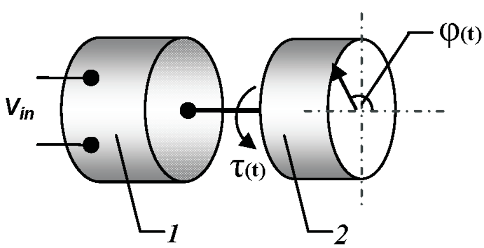

Figure 1. Principle scheme of Driveline

1-motor; 2-reduced load with moment of inertia J; $\varphi_{(t)}$-rotation angle; $\tau_{(t)}$-torque; V_{in}-input voltage

Motion of the system shown in Fig.1 is described by the following system of differential equations

$$\dot{\omega} = \frac{1}{J}\left(\sum_{i=1}^{n} M_i + \sum_{j=1}^{k} m_i \right) \tag{1}$$

$$\alpha = \dot{\varphi},$$

where ω is the angular rate of the controlled object under consideration; M_i (i=1, 2, …, n) are the uncontrolled Torques; m_j (j = 1, 2, …, k) are the controlled Torques.

Uncontrolled Torques may include, for example, all perturbations generated by the environment in which the motion takes place.

The terminal state control problem is formulated as follows: Given the initial phase state of the object ($\varphi_0; \dot{\varphi}_0$), it is required to transfer it (within time T) to the terminal state ($\varphi_f; \dot{\varphi}_f$). Below we discuss the solution of the formulated problem.

Synthesis of the General Control Function for Terminal Problems

Uncontrolled forces are functions of time t, the coordinate φ(angle) and velocity $\dot{\varphi}$ (angular rate): $M_i = M_i(t, \varphi, \dot{\varphi})$, while controlled forces, in addition to being all these functions, are also functions of the controlling parameter α[1]: $m_j = m_j(t, \varphi, \dot{\varphi}, \alpha)$. Note that the parameter α is frequently the position of the controlling element and may be a function of time. The traditional approach to the solution of the above-stated motion control problems consists in finding the functions $m_j = m_j(t, \varphi, \dot{\varphi}, \alpha)$ for which solutions of system (1) satisfy, on the time interval [0; T], the corresponding boundary conditions. As has been said, the uniqueness of a solution is obtained by using an additional condition that solutions must supply an extremum to some specially chosen functional. Such an additional condition is frequently the requirement for a control time minimum (quick action maximum) or an energy minimum of controlling forces. There are also other kinds of

[1] For example, in the case of jet engines the throttle quadrant may play the role of a controlling parameter.

functionals. Solutions obtained in this manner are of program character (the control system is open), which leads to the instability of the realized motion because of the unforeseen influence of uncontrolled forces. The development of an adaptive method demands a different approach: it is necessary to keep a continuous control over the current state of the controlled object and these demands to take respective measurements.

Let us discuss this issue in more detail. Let an optimal function (t) of controlling forces be defined in some manner. Then it is obvious that the controlling parameter function $\alpha(t)$ can be defined as a solution of some differential equation, the right-hand side of which depends on a difference between the given optimal function (t) of controlling forces and the current measured value of the resultant of these forces $m = m(t, \varphi, \dot{\varphi}, \alpha(t))$. Assume that this differential equation has the form

$$\dot{\alpha}(t) = k_c(\psi(t) - m(t, \varphi, \dot{\varphi}, \alpha)) \; [2]. \quad (2)$$

Assume that a relation between the controlling parameter $\alpha(t)$ and the value of the current (measured) force $m = m(t, \varphi, \dot{\varphi}, \alpha(t))$ can be written in the form of an inertia element of first order

$$\dot{m} = (k_f \alpha(t) - m). \quad (3)$$

The device described by equation (3) is a regulator, i.e. a power unit generating the controlling force $m = m(t, \varphi, \dot{\varphi}, \alpha(t))$.

The control process is therefore described by means of the system of differential equations (1) (3). Knowing the synthesized function of controlling forces (t), we can transfer the object from the initial state $\varphi(t_o); \dot{\varphi}(t_0)$ to the terminal state $\varphi(t_f); \dot{\varphi}(t_f)$. However here we encounter a difficulty caused by the necessity to measure controlling forces. This, obviously, can be done if these forces are separated from controlled forces during the object motion. From the practical standpoint, the latter is an unsolvable problem and this circumstance impedes the development of adaptive methods which could be applicable to problems of terminal state control.

[2] This control law gives astatism of third order with respect to external perturbing forces.

The problem we consider here can be solved by taking a different approach (Batenko, 1977, Erguven, 2004).

A change of controlling forces brings about a change of uncontrolled forces too. All forces (uncontrolled +controlled) acting on the controlled object generate the object motion acceleration $\dot{a}$. It is obvious that $\dot{a}$ can be easily measured directly and therefore we should pose the problem on the synthesis of a controlling function in the form of acceleration $\ddot{\varphi}(t)$. Then the control process reduces to the fulfillment of the equality

$$\dot{a} = \ddot{\varphi}(t), \tag{4}$$

where $\dot{a}$ is the measured acceleration of the object and $\ddot{\varphi}(t)$ is the desired (synthesized) acceleration of the object.

Note that (4) is actually the equation of motion of the controlled object under the action of the controlling function $\ddot{\varphi}(t)$ and is equivalent to (1). This is explained by the fact that the measured acceleration of the object $\dot{a}$ takes into account changes of both uncontrolled and controlled forces. We will make an essential use of this fact in the sequel. It is not difficult to realize equality (4) physically if the regulator (power unit) described by the equation of an inertia element (3) is sufficiently powerful. In that case it becomes possible to compensate uncontrolled forces by controlled ones and to fulfill equality (4).

Let us assume that the relation between the given acceleration $\ddot{\varphi}(t)$ and controlling forces $m = m(t, \varphi, \dot{\varphi}, \alpha(t))$ is

$$\ddot{\varphi} = km(t, \varphi, \dot{\varphi}, \alpha(t)) \tag{5}$$

where k is the proportionality coefficient.

The synthesis of a control algorithm can be reduced to some variational problem in a phase space: Given two points $(\varphi_o; \dot{\varphi}_0)$ and $(\varphi_f; \dot{\varphi}_f)$ in a two-dimensional phase space, it is required to derive the equation of a curve of this phase space that connects $(\varphi_o; \dot{\varphi}_0)$ and $(\varphi_f; \dot{\varphi}_f)$ and delivers a minimum to the next functional

$$J_f = \frac{1}{T}\int_0^T m^2(t,\varphi,\dot{\varphi},\alpha(t))dt\,. \tag{6}$$

The equation of the curve we want to define can be written parametrically as $\phi = \phi(t)$ and $\dot{\phi} = \dot{\phi}(t)$. Then it is obvious that to the phase curve defined in this manner there corresponds the motion trajectory from the point φ_0 to the point ϕ_f. The initial velocity at the initial moment of time $t = t_0$ is equal to $\dot{\varphi}_0$ and at the terminal moment of time $t = T$ -- to $\dot{\varphi}_f$.

From (6) it follows that the trajectory $\phi = \phi(t)$ and $\dot{\phi} = \dot{\phi}(t)$ delivering a minimum to (6) is optimal in the sense that it minimizes energetic controlling actions.

The acceleration along the optimal trajectory is the function of phase coordinates

$$\ddot{\varphi} = \Psi(\varphi,\dot{\varphi}). \tag{7}$$

From (5) and (6) we have

$$km(t,\varphi,\dot{\varphi},\alpha(t)) = \Psi(\varphi,\dot{\varphi}). \tag{8}$$

Substituting (8) into (6) we obtain

$$J = \frac{1}{T}\int_0^T k_1[\Psi(\varphi,\dot{\varphi})]^2\,dt = \frac{1}{T}\int_0^T [k_1\ddot{\varphi}]^2\,dt\,, \tag{9}$$

where $k_1 = \dfrac{1}{k}$.

Functional (9) belongs to the type of functionals containing derivatives of second order and therefore its corresponding Euler equation can be written in the form(Gelfand, 1961)

$$\frac{d^2\ddot{\varphi}}{dt^2} = 0\,. \tag{10}$$

Solution (11) is a third order polynomial

$$\varphi = C_0 + C_1 t + C_2 \frac{t^2}{2} + C_3 \frac{t^3}{6} \ . \tag{11}$$

Function (11) is result of synthesis of the most general Control function of Drivelines control process, which covers various particular control problems. These problems can be naturally defined by choosing of corresponding boundary conditions, which depict peculiarities of the each of the particular control problem.

In general the boundary conditions are:

$$t = 0; \ \varphi = \varphi_0; \ \dot{\varphi} = \dot{\varphi}_0 \ ; \tag{12}$$

$$t = T; \ \varphi = \varphi_f; \ \dot{\varphi} = \dot{\varphi}_f \ . \tag{13}$$

These four conditions are sufficient for defining four constants C_i (i = 0, 1, 2, 3) contained in (11), which completely defines an optimal trajectory.

As it was mentioned above by using initial conditions (12) and (13) it is possible to solve different terminal problems: reduction, acceleration breaking etc. We illustrate the terminal control approach with the example of only acceleration problem. One of the most important reasons of it is that the problem is very important for control especially of Drivelines (Geartrains).

The Approach Problem with an Additional Condition Imposed on the Terminal Accelerations

Frequently, it is not enough to have four boundary conditions (12) and (13) of the approach problem to solve applied problems of terminal control. For example, in the case deceleration it is not enough to assume that the terminal velocity is equal to zero: for a complete stop it is necessary that the terminal acceleration, too, be equal to zero. Thus there arise an additional boundary condition (the fifth one) related to acceleration:

$$t = 0; \ \varphi = \varphi_0; \ \dot{\varphi} = \dot{\varphi}_0 \ ;$$

$$t = T;\ \varphi = \varphi_f;\ \dot{\varphi} = \dot{\varphi}_f;\ \ddot{\varphi} = \ddot{\varphi}_f. \quad (14)$$

It is clear that in this case the controlling function should be taken in the form of a polynomial of fourth order containing five coefficients, of which only three are to be defined, since it is obvious that the first two coefficients satisfy the first two (initial) conditions (14)

$$\varphi(t) = \varphi_0 + \dot{\varphi}_0 t + C_2 t + C_3 t^2 + C_4 t^3 + C_5 t^4 . \quad (15)$$

Calculating the first and second derivatives and substituting them into (14), we obtain the values of the coefficients C_i (i = 2, 3, 4)

$$C_2 = \frac{12}{T^2}(\varphi_f - \varphi_0) - \frac{6}{T}(\dot{\varphi}_f + \dot{\varphi}_0) + \ddot{\varphi}_f;$$

$$C_3 = \frac{48}{T^3}(\varphi_f - \varphi_0) + \frac{18}{T^2}(\dot{\varphi}_f + \dot{\varphi}_0) - \frac{6}{T}\ddot{\varphi}_f;$$

$$C_4 = \frac{36}{T^4}(\varphi_f - \varphi_0) - \frac{12}{T^3}(\dot{\varphi}_f + \dot{\varphi}_0) + \frac{6}{T^2}\ddot{\varphi}_f. \quad (16)$$

From (15) and (16) it follows that the controlling acceleration function has the form

$$\ddot{\varphi}(t) = \frac{12}{T^2}(\varphi_f - \varphi_0) - \frac{6}{T}(\dot{\varphi}_f + \dot{\varphi}_0) + \ddot{\varphi}_f + (\frac{48}{T^3}(\varphi_f - \varphi_0) + \frac{18}{T^2}(\dot{\varphi}_f + \dot{\varphi}_0) - \frac{6}{T}\ddot{\varphi}_f)t +$$
$$+(\frac{36}{T^4}(\varphi_f - \varphi_0) - \frac{12}{T^2}(\dot{\varphi}_f + \dot{\varphi}_0) + \frac{6}{T^2}\ddot{\varphi}_f)t^2 .$$

(17)

To pass to the control with feedback we proceed as in the preceding cases, i.e. we assume that in (17) t = 0,   T = T – t, and replace the initial values of the phase coordinates by the respective current ones. As a result, we obtain.

$$\ddot{\varphi}(t) = \frac{12}{(T-t)^2}(\varphi_f - \varphi_0) - \frac{6}{(T-t)}(\dot{\varphi}_f + \dot{\varphi}); \quad (18)$$

(18) is again the law of control with a singularity and to eliminate this singularity we proceed as follows. We assume that T – t =   T =

const and the terminal values of the phase trajectories are equal to the variable phase trajectories of the mobile target point

$$\varphi_m(t) = \varphi_0 + \dot{\varphi}_{10}(t+\Delta T) + C_2\frac{(t+\Delta T)^2}{2} + C_3\frac{(t+\Delta T)^3}{6} + C_4\frac{(t+\Delta T)^4}{24},$$

$$\dot{\varphi}_m(t) = \dot{\varphi}_{10} + C_2(t+\Delta T) + C_3\frac{(t+\Delta T)^2}{2} + C_4\frac{(t+\Delta T)^3}{6}, \tag{20}$$

where C_2, C_3 and C_4 are defined from (14).

Substituting functions (20) into (18) and performing simple but rather lengthy transformations, we obtain the differential equation of the approach problem which does not contain singularities

$$\ddot{\varphi} + K_\omega + K_\varphi \varphi = \sum_{i=0}^{4} K_i t^i, \tag{21}$$

where

$$K_0 = \frac{12\varphi_0}{\Delta T^2} + \frac{6\dot{\omega}_0}{\Delta T} + C_2;$$

$$K_1 = \frac{12\dot{\varphi}_0}{\Delta T^2} + \frac{6C_2}{\Delta T} + C_3;$$

$$K_2 = \frac{6C_2}{\Delta T^2} + \frac{3C_3}{\Delta T} + C_4;$$

$$K_3 = \frac{2C_3}{\Delta T^2} + \frac{2C_4}{\Delta T};$$

$$K_4 = \frac{C_4}{\Delta T^2};$$

$$K_\varphi = \frac{12}{\Delta T^2};$$

$$K_\omega = \frac{6}{\Delta T}.$$

Let us define the transitional and stationary components of equation (21). A particular solution of the non-homogeneous equation will be sought in the form

$$\varphi = \sum_{i=0}^{4} a_i t^i, \qquad (22)$$

where a_i are the coefficients we want to define.

Differentiating (22) twice, substituting into (21) and equating the coefficients at equal powers t, we obtain a system of equations with respect to the desired coefficients a_i ($i = 0,\ldots,4$)

$$\begin{aligned} K_0 &= 2a_2 + K_\omega a_1 + K_\varphi a_0; \\ K_1 &= 6a_3 + 2a_2 K_\omega + a_1 K_\varphi; \\ K_2 &= 12a_4 + 3a_3 K_\omega + a_2 K_\varphi; \\ K_3 &= 4a_4 K_\omega + a_3 K_\varphi; \\ K_4 &= a_4 K_\omega, \end{aligned} \qquad (23)$$

from which they are defined quite easily:

$$\begin{aligned} a_4 &= \frac{K_4}{K_\varphi}; \\ a_3 &= \frac{K_3 - 4K_\omega a_4}{K_\varphi} \\ a_2 &= \frac{K_2 + 3K_\omega a_3 - 12a_4}{K_\varphi}; \\ a_1 &= \frac{K_1 - 2K_\omega a_2 - 6a_3}{K_\varphi}; \\ a_0 &= \frac{K_1 - K_\omega a_1 - 2a_2}{K_\varphi}. \end{aligned} \qquad (23)$$

Expressions (23) define the stationary component of the approach process with the given terminal (zero) acceleration value.

The transitional component (a general solution of the non-homogeneous equation (21)) is likewise easy to write:

$$\varphi_{tr}(t) = e^{-\frac{K_\omega t}{2}}(c_1 cos(\beta t) + c_2 sin(\beta t)), \tag{24}$$

where

$\beta = \sqrt{K_\varphi - (\frac{K_\omega}{2})}$, c_1, c_2 are the constants we want to define.

A complete solution of the differential equation (21) can now be written as a sum of the transitional and the stationary process

$$\varphi(t) = \varphi_{tr}(t) + \varphi_{tr}(t) = e^{-\frac{K_\omega t}{2}}(c_1 cos(\beta t) + c_2 sin(\beta t)) + \sum_{i=0}^{4} a_i t^i . \tag{25}$$

To define the constants s_1 and s_2 we use the initial conditions (14) and derivative of (25), which gives the following expressions for the sought constants

$$C_1 = \varphi_{10} - a_0 \; ; \;\; C_2 = (\dot{\varphi}_{10} - a_1 K_\omega \frac{C_1}{2})\frac{1}{\beta} \tag{26}$$

and eventually the final expression for a complete solution of (21).

$$\varphi(t) = e^{-\frac{K_\omega t}{2}}((\varphi_{10} - a_0)cos(\beta t) + (\dot{\varphi}_{10} - a_1 K_\omega \frac{(\varphi_{10} - a_0)}{2})sin(\beta t)) + \sum_{i=0}^{4} a_i t^i . \tag{27}$$

The transitional process (24) gets damped with time (the time constant is equal to $\frac{K_\omega}{2}$), i.e. the object moves to the forced trajectory (22).

The velocity of the controlled object is equal to

$$\dot{\varphi}(t) = e^{\frac{K_\omega}{2}}\left(-\frac{K_\omega}{2}(\varphi_{10} - a_0)cos(\beta t) + (\omega_{10} - a_1 K_\omega \frac{(\varphi_{10} - a_0)}{2})sin(\beta t)\right) +$$
$$+\left((\varphi_{10} - a_0)\beta sin(\beta t) + (\omega_{10} - a_1 K_\omega \frac{(\varphi_{10} - a_0)}{2})\beta cos(\beta t) \right) +$$
$$+ a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 . \tag{28}$$

Substituting the value t = T into the stationary solution of equation (21) and into its derivative, it is not difficult to see that they indeed satisfy the boundary conditions (14) provided that the terminal acceleration is equal to zero, which solves the posed problem on the terminal state control in the approach problem.

Conclusion

General Principles of Drivelines Terminal Control are suggested. The terminal state control problem applied to Drivelines Control is defined. It is shown that on the base of Driveline acceleration measurement it is possible to synthesize general Control function of Drivelines control process. The last function covers various particular control problems. It is shown, that these problems can be naturally defined by choosing of corresponding boundary conditions. As an example of such particular cases an Approach problem with an Additional Condition Imposed on the Terminal Accelerations is solved.

References

Batenko (1977), *Terminal Control of Moving Objects,* Moscow, Sovetskoe Radio, (in Russsian), 256

Erguven J., Milnikov A.A.,. Rodonaia I.D., Suladze A.S. (2004), “Moving Mechanical Objects Terminal Adaptive Control Problems”, *Problems of Applied Mechanics*: Tbilisi, №2(15), p.54-58

Gelfand I.M., Fomin S.V. (1961), *Variational Calculus*, Moscow, Pismatgiz, (in Russsian), 228

Mchedlishvili T.F. (2008), *Scientific foundation and applied problems of nonlinear Drivelines systems synthesis according to given transient processes* (in Russsian), Tbilisi, Georgian Technical University Press, 273

Petrov I.P (1971), *Optimal Control of Electro Drivelines,* Leningrad, Energia, 345