Synthesis of Control of Terminal Acceleration of Spatial Rotations

Cabir ERGÜVEN

Abstract: *Among the problems of Adaptive Terminal Control, one of the most important is the problem of control of acceleration process of moving objects. Two cases are discussed: Control with arbitrary terminal acceleration and control with certain value of terminal acceleration. The control law functions are obtained and analysis of the both process is represented. The obtained results can be used for practical purposes to elaborate simple control algorithms.*

Keywords: Terminal Control, Moving Objects, Boundary Problems, Adaptive Control, Reduction, Acceleration, Transient Process.

Introduction

The problem of terminal acceleration can be met often in spatial movement control theory and in practice as well [Krasovski, 1971]. The importance of the problem comes from the fact that configuration coordinates and velocities are not enough to define terminal state completely [Batenko A.P. (1977)].

The problem of control of terminal acceleration can be divided into two sub-problems:

1. Control with arbitrary terminal acceleration
2. Control with certain value of terminal acceleration [Krasovski, 1971]

Control with Arbitrary Terminal Acceleration (Approach Problem)

Synthesis of the Control

The approach problem employs four boundary conditions t = 0;

$\gamma = \gamma_0$; $\dot{\gamma} = \dot{\gamma}_0$ and $t=T$; $\gamma = \gamma_f$; $\dot{\gamma} = \dot{\gamma}_f$ which allow us to calculate

Cabir ERGÜVEN *is an associate professor in Faculty of Computer Technologies and Engineering at International Black Sea University*

immediately the coefficients C_i (i=0, 1, 2, 3) in the controlling function

$$\gamma = C_0 + C_1 t + C_2 \frac{t^2}{2} + C_3 \frac{t^3}{6}. \qquad C_0 = \gamma_0; C_1 = \dot{\gamma}_0; \qquad (1)$$

$$C_2 = \frac{6}{T^2}(\gamma_f - \gamma_0) - \frac{2}{T}(2\dot{\gamma}_f + \dot{\gamma}_0); C_3 = \frac{12}{T^3}(\gamma_0 - \gamma_f) - \frac{6}{T^2}(\dot{\gamma}_f + \dot{\gamma}_0). \quad (2)$$

Since for the acceleration (1) implies

$$\ddot{\gamma}(t) = C_2 + C_3 t, \qquad (3)$$

we obtain the synthesized control function

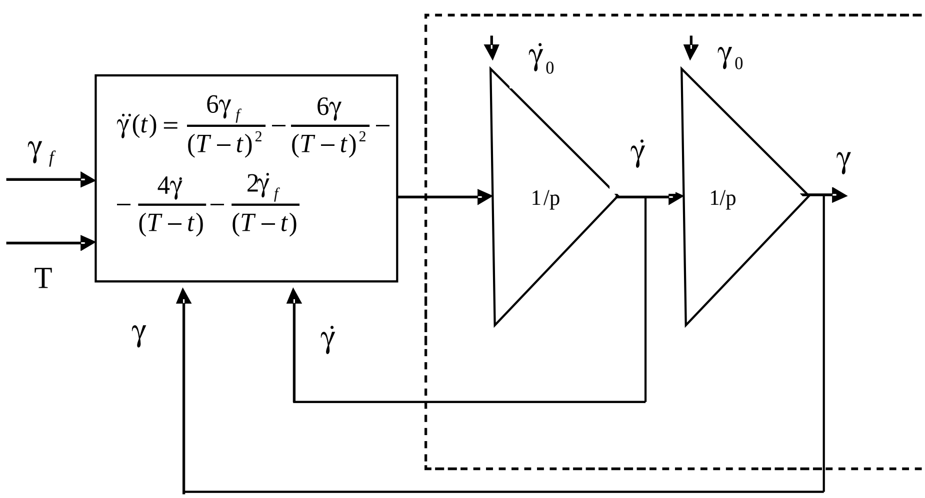
$$\ddot{\gamma}(t) = (\frac{6}{T^2}(\gamma_f - \gamma_0) - \frac{2}{T}(2\dot{\gamma}_f + \dot{\gamma}_0)) + (\frac{12}{T^3}(\gamma_0 - \gamma_f) - \frac{6}{T^2}(\dot{\gamma}_f + \dot{\gamma}_0))t \qquad (4)$$

of the velocity and coordinate program

$$\dot{\gamma}(t) = \dot{\gamma}_0 + (\frac{6}{T^2}(\gamma_f - \gamma_0) - \frac{2}{T}(2\dot{\gamma}_f + \dot{\gamma}_0))t + (\frac{12}{T^3}(\gamma_0 - \gamma_f) - \frac{6}{T^2}(\dot{\gamma}_f + \dot{\gamma}_0))\frac{t^2}{2} \qquad (5)$$

$$\gamma(t) = \gamma_0 + \dot{\gamma}_0 t + (\frac{6}{T^2}(\gamma_f - \gamma_0) - \frac{2}{T}(2\dot{\gamma}_f + \dot{\gamma}_0))\frac{t^2}{2} + (\frac{12}{T^3}(\gamma_0 - \gamma_f) - \frac{6}{T^2}(\dot{\gamma}_f + \dot{\gamma}_0))\frac{t^3}{6} \qquad (6)$$

Figure 1 The block-diagram of the approach program



In order to obtain an adaptive control algorithm we proceed as follows: since now the object is all the time at the initial point of time, it is assumed that t=0 and the initial velocity and coordinate values are replaced by the respective current values, and the moment of time T is replaced by the difference $T-t$:

$$\ddot{\gamma}(t)=\frac{6\gamma_f}{(T-t)^2}-\frac{6\gamma}{(T-t)^2}-\frac{4\dot{\gamma}}{(T-t)}-\frac{2\dot{\gamma}_f}{(T-t)} \qquad (7)$$

Since control $\ddot{\gamma}=\frac{\dot{\gamma}_f-\dot{\gamma}}{T-t}$ again contains the same singularity as the law of adaptive control of the reduction process, we should use an analogous method of its elimination.

Analysis of the Control Process Dynamics

In (7), replace $T-t$ by ÄT, where ÄT is a constant time interval, i.e. it is again assumed that the target point of the approach process is mobile. Its variable coordinate denote by γ_m is obviously equal to

$$\gamma(t)=\gamma_0+\dot{\gamma}_0(t+\Delta T)+\left(\frac{6}{T^2}(\gamma_f-\gamma_0)-\frac{2}{T}(2\dot{\gamma}_f+\dot{\gamma}_0)\right)\frac{(t+\Delta T)^2}{2}+$$
$$+\left(\frac{12}{T^3}(\gamma_0-\gamma_f)-\frac{6}{T^2}(\dot{\gamma}_f+\dot{\gamma}_0)\right)\frac{(t+\Delta T)^3}{6} \qquad (8)$$

It is easy to see that the velocity of the mobile target point is equal to

$$\dot{\gamma}(t)=\dot{\gamma}_0+\left(\frac{6}{T^2}(\gamma_f-\gamma_0)-\frac{2}{T}(2\dot{\gamma}_f+\dot{\gamma}_0)\right)(t+\Delta T)+\left(\frac{12}{T^3}(\gamma_0-\gamma_f)-\right.$$
$$\left.-\frac{6}{T^2}(\dot{\gamma}_f+\dot{\gamma}_0)\right)\frac{(t+\Delta T)^2}{2}. \qquad (9)$$

Substituting (8), (9) and $T-t=$ ÄT into (7) and performing some transformations, we obtain the differential equation of second order

$$\ddot{\gamma}+K_v\dot{\gamma}+K_s\gamma=K_0+K_1t+K_2t^2+K_3t^3 \qquad (10)$$

where

$$K_0=\frac{6\gamma_0}{\Delta T^2}+\frac{4\dot{\gamma}_0}{\Delta T}+C_2 \quad K_1=\frac{6\dot{\gamma}_0}{\Delta T^2}+\frac{4C_2}{\Delta T}+C_3 \quad K_2=\frac{3C_2}{\Delta T^2}+\frac{2C_3}{\Delta T}$$

$$K_s = \frac{6}{\Delta T^2} \qquad K_v = \frac{4}{\Delta T}$$

C2 and C3 are defined from (2).

The forced component from the general solution (10) has the form

$$\gamma_{fr} = \frac{\Delta T^2}{6}[K_0 - \frac{2}{3}\Delta T K_1 + \frac{5}{9}\Delta T^2 K_2 - \frac{4}{9}\Delta T^3 K_3 + \left(K_1 - \frac{4}{3}\Delta T K_2 + \frac{5}{3}\Delta T^2 K_3\right)t + $$
$$+ \left(K_2 - 2\Delta T K_3\right)t^2 + K_3 t^3] \qquad (11)$$

The transitional component [Batenko A.P. (1977), Milnikov (2004)] is written as follows:

$$\gamma_{tr} = e^{-\frac{2t}{\Delta T}}\left[\gamma_{10}\left(\cos\frac{\sqrt{2}}{\Delta T} + \sqrt{2}\sin\frac{\sqrt{2}}{\Delta T}t\right) + \dot{\gamma}_{10}\frac{\sqrt{2}}{2}\Delta T\sin\frac{\sqrt{2}}{\Delta T}t - \frac{\Delta T^2}{6}K_0\left(\cos\frac{\sqrt{2}}{\Delta T}t + \sqrt{2}\sin\frac{\sqrt{2}}{\Delta T}t\right) + \right.$$
$$+ \frac{\Delta T^3}{9}K_1\left(\cos\frac{\sqrt{2}}{\Delta T}t + \frac{\sqrt{2}}{4}\sin\frac{\sqrt{2}}{\Delta T}t\right) - \frac{5}{54}\Delta T^4 K_2\left(\cos\frac{\sqrt{2}}{\Delta T}t - \frac{\sqrt{2}}{5}\sin\frac{\sqrt{2}}{\Delta T}\right) +$$
$$\left. + \frac{\sqrt{2}}{54}\Delta T^5 K_3\left(2\sqrt{2}\cos\frac{\sqrt{2}}{\Delta T}t - \frac{t}{2}\sin\frac{\sqrt{2}}{\Delta T}t\right)\right] =$$
$$= e^{-\frac{2t}{\Delta T}}\left(A\cos\frac{\sqrt{2}}{\Delta T}t + B\sin\frac{\sqrt{2}}{\Delta T}t\right) \qquad (12)$$

where

$$A = \gamma_{10} - \frac{1}{6}\Delta T^2 K_0 + \frac{1}{9}\Delta T^3 K_1 - \frac{5}{54}\Delta T^4 K^2 + \frac{4}{54}\Delta T^5 K_3$$

$$B = \sqrt{2}\gamma_{10} + \frac{\sqrt{2}}{2}\dot{\gamma}_{10}\Delta T - \frac{\sqrt{2}}{6}\Delta T^2 K_0 + \frac{\sqrt{2}}{36}\Delta T^3 K_1 + \frac{5\sqrt{2}}{270}\Delta T^4 K_2 - \frac{7\sqrt{2}}{108}\Delta T^5 K_3 \qquad (13)$$

It should be emphasized that in the above expressions the initial values γ_{10} *and* $\dot{\gamma}_{10}$ are not equal to the initial values given t = 0; $\gamma = \gamma_0$; $\dot{\gamma} = \dot{\gamma}_0$ and thus there arises the transitional process (12) which gets damped with time (in this case the time constant is equal to Ä*T*/2 , i.e. the object moves to the forced trajectory (11), which leads to a complete solution of the approach problem.

The Problem with an Additional Condition Imposed on the Terminal Accelerations

Synthesis of the Control

Frequently, it is not enough to have four boundary conditions $t = 0$; $\gamma = \gamma_0$; $\dot\gamma = \dot\gamma_0$, and $t = T$; $\gamma = \gamma_f$; $\dot\gamma = \dot\gamma_f$ of the approach problem to solve applied problems of terminal control. For example, in the case of deceleration it is not enough to assume that the terminal velocity is equal to zero: for a complete stop it is necessary that the terminal acceleration, too, be equal to zero. Thus, there arise an additional boundary condition (the fifth one) related to acceleration:

$$t=0;\ \gamma = \gamma_0;\ \dot\gamma = \dot\gamma_0, \qquad t=T;\ \gamma = \gamma_f;\ \dot\gamma = \dot\gamma_f; \ddot\gamma = \ddot\gamma_f \tag{13}$$

It is clear that in this case the controlling function should be taken in the form of a polynomial of fourth order containing five coefficients, of which only three are to be defined, since it is obvious that the first two coefficients satisfy the first two (initial) conditions (13)

$$\gamma(t) = \gamma_0 + \dot\gamma_0 t + C_2 t + C_3 t^2 + C_4 t^3 + C_5 t^4 \tag{14}$$

Calculating the first and second derivatives and substituting them into the last three equations (13),

we obtain the values of the coefficients C_i (i = 2, 3, 4)

$$C_2 = \frac{12}{T^2}(\gamma_f - \gamma_0) - \frac{6}{T}(\dot\gamma_f + \dot\gamma_0) + \ddot\gamma_f \quad ;$$

$$C_3 = \frac{48}{T^3}(\gamma_f - \gamma_0) + \frac{18}{T^2}(\dot\gamma_f + \dot\gamma_0) - \frac{6}{T}\ddot\gamma_f \, ;$$

$$C_4 = \frac{36}{T^4}(\gamma_f - \gamma_0) - \frac{12}{T^2}(\dot\gamma_f + \dot\gamma_0) + \frac{6}{T^2}\ddot\gamma_f \, . \tag{15}$$

From (14) and (15) it follows that the controlling acceleration function has the form

$$\ddot\gamma(t) = \frac{12}{T^2}(\gamma_f - \gamma_0) - \frac{6}{T}(\dot\gamma_f + \dot\gamma_0) + \ddot\gamma_f + \left(\frac{48}{T^3}(\gamma_f - \gamma_0) + \frac{18}{T^2}(\dot\gamma_f + \dot\gamma_0) - \frac{6}{T}\ddot\gamma_f\right)t +$$

$$+\left(\frac{36}{T^4}(\gamma_f - \gamma_0) - \frac{12}{T^2}(\dot\gamma_f + \dot\gamma_0) + \frac{6}{T^2}\ddot\gamma_f\right)t^2. \tag{16}$$

Analysis of the Control Process Dynamics

To pass to the control with feedback we proceed as in the preceding cases, i.e. we assume that in (16) $t = 0$, Ä$T = T - t$, and replace the initial values of the phase coordinates by the respective current ones. As a result, we obtain

$$\ddot{\gamma}(t) = \frac{12}{(T-t)^2}\left(\gamma_f - \gamma\right) - \frac{6}{(T-t)}\left(\dot{\gamma}_f + \dot{\gamma}\right)^{1}. \qquad (17)$$

16) is again the law of control with a singularity and to eliminate this singularity we proceed as before. We assume that $T - t =$ Ä$T =$ const and the terminal values of the phase trajectories are equal to the variable phase trajectories of the mobile target point

$$\gamma_m(t) = \gamma_0 + \dot{\gamma}_{10}(t+\Delta T) + C_2\frac{(t+\Delta T)^2}{2} + C_3\frac{(t+\Delta T)^3}{6} + C_4\frac{(t+\Delta T)^4}{24}$$

$$\dot{\gamma}_m(t) = \dot{\gamma}_{10} + C_2(t+\Delta T) + C_3\frac{(t+\Delta T)^2}{2} + C_4\frac{(t+\Delta T)^3}{6} \qquad (18)$$

where C_2, C_3 and C_4 are defined from (15).

Substituting functions (18) into (17) and performing simple but rather lengthy transformations, we obtain the differential equation of the approach problem which does not contain singularities

$$\ddot{\gamma} + K_\omega\dot{\gamma} + K_\gamma\gamma = \sum_{i=0}^{4} K_i t^i \qquad (19)$$

$$K_0 = \frac{12\gamma_0}{\Delta T^2} + \frac{6\omega_0}{\Delta T} + C_2\,; \quad K_1 = \frac{12\omega_0}{\Delta T^2} + \frac{6C_2}{\Delta T} + C_3\,; \quad K_2 = \frac{6C_2}{\Delta T^2} + \frac{3C_3}{\Delta T}C_4;$$

$$K_2 = \frac{2C_3}{\Delta T^2} + \frac{2C_4}{\Delta T}\,; \quad K_4 = \frac{C_4}{\Delta T^2}\,; \quad K_\gamma = -\frac{12}{\Delta T^2}\,; \quad K_\omega = -\frac{6}{\Delta T}\,;$$

Let us define the transitional and stationary components of equation (19). A particular solution of the non-homogeneous equation will be sought in the form $\gamma = \sum_{i=0}^{4} a_i t^i$ where a_i are the coefficients we want to define.

[1] The terminal acceleration value is assumed to $\ddot{\gamma}_f = 0$, which is natural for the deceleration (stopping) problem.

Differentiating (20) twice, substituting into (19) and equating the coefficients at equal powers t, we obtain a system of equations with respect to the desired coefficients a_i (i = 0,…,4)

$$K_o = 2a_2 + K_\omega a_1 + K_\gamma a_0; K_1 = 6a_3 + 2a_2\ K_\omega + a_1 K_\gamma;$$

$$K_2 = 12a_4 + 3a_3\ K_\omega + a_2 K_\gamma\ ; K_3 = 4a_4 K_\omega + a_3 K_\gamma; K_4 = a_4\ K_\omega \quad (21)$$

from which they are defined quite easily:

$$a_4 = -\frac{K_4}{K_\gamma}; a_2 = \frac{K_2 + 3K_w a_3 - 12a_4}{K_\gamma}; a_1 = \frac{K_1 + 2K_w a_2 - 6a_3}{K_\gamma}; a_0 = \frac{K_w a_1 - 2a_2}{K_\gamma} \quad (22)$$

Expressions (22) define the stationary component of the approach process with the given terminal (zero) acceleration value.

The transitional component (a general solution of the non-homogeneous equation (19)) is likewise easy to write:

$$\gamma_{tr}(t) = e^{-\frac{K_w t}{2}} (s_1 \cos\beta t + s_2 Sin\beta t) \quad (23)$$

where $\beta = \sqrt{k_\gamma - \left(\frac{k_\omega}{2}\right)^2}$, s_1, s_2 are the constants we want to define.

A complete solution of the differential equation (19) can now be written as a sum of the transitional and the stationary process

$$\gamma(t) = \gamma_{tr}(t) + \gamma_{tr}(t) = e^{-\frac{K_\omega t}{2}} (s_1 \cos(\beta t) + s_2 \sin\beta t) + \sum_{i=0}^{4} a_i t^i \quad (24)$$

To define the constants s1 and s2 we use the initial conditions t = 0; $\gamma = \gamma_{10}$; $\dot\gamma = \dot\gamma_{10}$ and derivative (24), which gives the following expressions for the sought constants

$$C_1 = \gamma_{10} - a_0; \quad C_2 = \left(\dot\gamma_{10} - a_1 K_\omega \frac{C_1}{2}\right)\frac{1}{\beta} \quad (25)$$

and eventually the final expression for a complete solution of (19).

$$\gamma(t) = e^{-\frac{K_\omega t}{2}} ((\gamma_{10} - a_0)\cos\beta t + (\dot\gamma_{10} - a_1 K_\omega \frac{(\gamma_{10} - a_0)}{2})\sin\beta t) + \sum_{i=0}^{4} a_i t^i \quad (26)$$

The transitional process (23) gets damped with time (the time constant is equal to $\frac{K_\omega}{2}$),i.e. the object moves to the forced trajectory (20).

The velocity of the controlled object is equal to

$$\dot{\gamma}(t) = e^{\frac{K_\omega}{2}} \left(-\frac{K_\omega}{2}(\gamma_{10} - a_0)\cos\beta t + \left(\omega_{10} - a_1 K_\omega \frac{\gamma_{10-}a_0}{2}\right)\sin\beta t + \right. \qquad (27)$$

$$+\left((\gamma_{10} - a_0)\beta sisn\beta t + \left(\omega_{10} - a_1 K_\omega \left(\frac{\gamma_{10-}a_0}{2}\right)\right)\beta\cos\beta t\right) + a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3$$

Substituting the value $t = T$ into the stationary solution of equation (19) and into its derivative, it is not difficult to see that they indeed satisfy the boundary conditions $t = T$; $\gamma = \gamma_f$; $\dot{\gamma} = \dot{\gamma}_f$ provided that the terminal acceleration is equal to zero, which solves the posed problem on the terminal state control in the approach problem.

References

1. Krasovski N.N. The Theory of moving control. Moscow, Nauka, 1971 (Красовский Н.Н. Теория управления движением. М: Наука, 1971)

2. Batenko A.P. (1977) Control of terminal state of moving objects, Moscow, Sovetskoe Radıo (Батенко А.П. Управление конечным состоянием движущихся объектов. М.: Советское Радио, 1977, 256 с.

3. Erguven C., Milnikov A.A.,. Rodonaia I.D., Suladze A.S. (2004), Moving Mechanical Objects Terminal Adaptive Control Problems // Problems of Applied Mechanics, Tbilisi, №2(15), p.54-58

4. Migulin V. V., (1972) Foundations of Oscilation Theory p.391 (Мигулин В.В. и др. Основы теории колебаний. М.: Наука, 1978, 391с.)

5. “Mechanical Engineering”, (1978) Vibration in Engineering, V1, , p.352 (Вибрации в технике т.1. М.: машиностроение, 1978, 352с.)