The singular operator and the Riesz potential operator in the Lebesgue Spaces with Variable Exponent on the real line

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Abstract

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1 Introduction

We consider the singular integral operator

$$Sf(x) : = \frac{1}{\pi} \int_{\mathbb{R}^1} \frac{f(y)}{y - x} \, dy, \quad x \in \mathbb{R}^1, \quad (1.1)$$

and the Riesz potential operator

$$P^\alpha f(x) : = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^1} \frac{f(y)}{y} \, \frac{dy}{|x|^{1-\alpha}}, \quad x \in \mathbb{R}^1, \quad (1.2)$$

within the frameworks of weighted spaces $L^{p(x)}(\mathbb{R}^1)$ with variable exponent $p(x)$. We refer, for example, to [9], [12], [10] for the Lebesgue spaces with variable exponent.

The progress in boundedness results for singular (and maximal) operators and for potential type operators in the spaces $L^{p(x)}(\Omega), \ \Omega \subset \mathbb{R}^1$ is mainly...
related to the case of bounded domains \( \Omega \). For unbounded domains we note an important result in [1] which provides the boundedness of the maximal operator in the space \( L^{p(x)}(\mathbb{R}^n) \) under the natural assumptions on \( p(x) \):

\[
1 < p_0 \leq p(x) \leq p < \infty, \quad x \in \mathbb{R}^n \tag{1.3}
\]

\[
|p(x) - p(y)| < A \frac{1}{|x - y|}, \quad x, y \in \mathbb{R}^n, \quad x \neq y \tag{1.4}
\]

and there exists \( \lim_{x \rightarrow \infty} p(x) - p(\infty) \) and

\[
p(x) - p(\infty) \leq A \ln(1 + |x|), \quad x, y \in \mathbb{R}^n. \tag{1.5}
\]

The Sobolev theorem for the Riesz potential operator was proved in [11], [6]-[5] for bounded domains and in [2] on the whole space \( \mathbb{R}^n \) under the assumption that \( p(x) = \text{const} \) outside some large ball. In [7] Sobolev theorem was proved for \( p(x) \) non-necessarily constant at infinity, but with some "extra" weight fixed to infinity and under the assumption that \( \min_{x \in \mathbb{R}^n} p(x) = p(\infty) \).

Within the framework of unbounded domains the paper [3] is also relevant, in which, in particular, there was obtained a Hardy-type inequality on the half-axis.

Meanwhile, for unbounded domains the following boundedness problems still remain open:

1) weighted estimates for the Hardy maximal operator;
2) weighted estimates for singular operators;
3) weighted (and non-weighted) \( L^{p(x)} \), \( L^{p(x)} \)-estimates for the Riesz potential operator.

We treat problems 2) and 3) in the one-dimensional case \( n - 1 \) for operators on \( \mathbb{R}^1 \) showing that in this case their solution may be obtained from the known results for bounded domains.

2 Preliminaries.

We deal with the spaces \( L^{p(x)}(\mathbb{R}^1) \) with \( p(x) \) treated as function on \( \mathbb{R}^1 \) where \( \mathbb{R}^1 \) is the compactification of \( \mathbb{R}^1 \) by the unique infinite point. We assume that the function \( p(x) \) has the logarithmic smoothness property not only locally
as in (1.4), but also at infinity in the sense that the function \( p \left( \frac{1}{y} \right) \) has this property for small \( x \), that is,

\[
\left| p \left( \frac{1}{x} \right) - p \left( \frac{1}{y} \right) \right| \leq \frac{A_{\infty}}{m \cdot \frac{1}{x^2}}, \quad |x - y| \leq \frac{1}{2}, \quad |x| \leq 1, \quad |y| \leq 1. \tag{2.1}
\]

From (2.1) it follows that there exists the limit

\[
p(\infty) := \lim_{x \to \infty} p(x)
\]

By \( q(x) \) we denote the conjugate exponent, \( \frac{1}{r(x)} + \frac{1}{q(x)} = 1 \).

The equivalence \( f(x) \approx g(x) \) for non-negative functions \( f(x) \) and \( g(x) \) means that there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 f(x) \leq g(x) \leq C_2 f(x).
\]

Let \( C \) be a curve on the complex plane and

\[
r(t) = \prod_{k=1}^m \left| t - t_k \right|^{\mu_k}, \quad t_k \in C, \quad k = 1, 2, \ldots, m. \tag{2.2}
\]

In [6]–[3], the following theorem was proved.

**Theorem 2.1.** Let \( C \) be a Lyapunov curve (or a curve of bounded rotation without cusps) and let \( p(t) \) be a function defined on \( C \) which satisfies conditions (1.3) and (1.4) on \( C \). The operator

\[
S_r f(t) = r(t) \int_{\mathbb{R}} \frac{f(\tau)}{r(\tau)(t - \tau)} \, d\tau
\]

is bounded in the space \( L^{p(t)}(C) \) if and only if

\[
- \frac{1}{p(t_k)} < \mu_k < \frac{1}{q(t_k)}, \quad k = 1, 2, \ldots, m. \tag{2.4}
\]

The Sobolev Theorem for the spaces \( L^{p(t)}(\Omega) \) in the case of bounded domains in \( \mathbb{R}^1 \) runs as follows (see [11], Theorem 3.2 and [6]–[5], Theorem B).
Theorem 2.2. Let \( p(x) \) satisfy assumptions (1.3)-(1.4) and the function 
\( \alpha(x) : \Omega \rightarrow (0, a) \) satisfy the conditions
\[
\inf_{x \in \Omega} \alpha(x) > 0 \quad \text{and} \quad \sup_{x \in \Omega} \alpha(x) p(x) < a. \tag{2.5}
\]
Then the potential operator
\[
I^{\alpha(\cdot)} f(x) = \int_{\Omega} \frac{f(y)}{|x - y|^n \alpha(y)} \, dy \tag{2.6}
\]
is bounded from \( L^{p(\cdot)}(\Omega) \) into \( L^{p(\cdot)}(\Omega) \) with
\[
\frac{1}{p_{\alpha}(x)} - \frac{1}{p(x)} = \frac{1}{a} \alpha(x). \tag{2.7}
\]

From the statement of Theorem 2.2 for \( n = 1, \Omega = [0, t] \), the following result may be easily derived.

Corollary 2.3. Let \( C \) be a Lipschitz curve on the complex plane, \( p(t) \) satisfy conditions (1.3)-(1.4) on \( C \) and \( \alpha(t) : C \rightarrow (0, 1) \) satisfy the conditions
\( \inf_{t \in C} \alpha(t) > 0 \) and \( \sup_{t \in C} \alpha(t) p(t) < 1 \). Then the potential type operator
\[
I^{p(\cdot)} f(t) = \int_{C} \frac{f(\tau)}{|t - \tau|^{n-1} \alpha(t)} \, d\tau \tag{2.7}
\]
is bounded from the space \( L^{p(\cdot)}(C) \) into \( L^{p(\cdot)}(C) \) with
\[
\frac{1}{p_{\alpha}(t)} - \frac{1}{p(\cdot)} = \frac{1}{a} \alpha(t). \tag{2.8}
\]

In [13] (see Theorem A and Remark 3.1 in [13]), the following statement was proved, in which it is supposed that the variable order \( \alpha(x), x \in \Omega \) satisfies the assumptions
\[
\inf_{x \in \Omega} \alpha(x) > 0 \quad \text{and} \quad \max_{x \in \Omega} \alpha(x) < 1, \tag{2.9}
\]
and
\[
\alpha(x) - \alpha(y) \leq \frac{A}{\ln \frac{1}{|x - y|}}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \Omega. \tag{2.10}
\]

Theorem 2.4. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Suppose that \( p(x) \) satisfies conditions (1.3)-(1.4) on \( \Omega \). Then the weighted potential operator
\[
x - x_0 \delta - \alpha \int_{\Omega} \frac{f(y) \, dy}{|y - x_0|^{n-1} \alpha(y)}, \quad x_0 \in \Omega \tag{2.10}
\]

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is bounded in the space $L^{p(x)}(\Omega)$ if

$$\alpha(x_0) \frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}.$$ 

A similar statement in the case $n = 1$ and $\beta = 0$ was obtained in [3] (see Theorem 2.1 in [3]).

**Corollary 2.5.** Let $C$ be a Lyapunov curve on the complex plane and

$$\rho_j(t) = \prod_{k=1}^{m} |t - t_k|^\gamma_k \quad \text{and} \quad \rho_p(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}. \quad (2.11)$$

Suppose that $\alpha(t) : C \to (0, 1)$ satisfies assumption (1.2) on $C$ and $0 < \min_{t \in C} \alpha(t) < \max_{t \in C} \alpha(t) < 1$ and $p(t)$ satisfies assumptions (1.1)-(1.2). Then the operator

$$\rho_j(t) \int_C \rho_p(t)|t - i|^{1 - \alpha(t)}$$

is bounded in the space $L^{p(t)}(C)$ provided that

$$\gamma_k - \mu_k \alpha(t_k), \quad \alpha(t_k) \frac{1}{\rho_p(t_k)} < \mu_k < \frac{1}{q(t_k)}, \quad k = 1, 2, \ldots, m.$$ 

We shall also make use of the following result for the weighted maximal operator

$$M^\beta f(x) = |x - x_0|^\beta \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x) \cap \Omega} |f(y)| \frac{1}{y - x_0|^\beta} \, dy, \quad (2.12)$$

where $\Omega$ is a bounded open set in $\mathbb{R}^n$ and $x_0 \in \Omega$, proved in [8], [4].

**Theorem 2.6.** Let $p(x)$ satisfy conditions (1.3), (1.4). The operator $M^\beta$ is bounded in $L^{p(x)}(\Omega)$ if and only if

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}. \quad (2.13)$$
Corollary 2.7. Let $C$ be a smooth curve in the complex plane satisfying the arc-cord condition. The weighted maximal operator on $C$

$$M^* f(t) = \sup_{\varepsilon > 0} r(t) \int_{C(t, \varepsilon)} \frac{|f(\tau)|}{r(\tau)} \, |d\tau|,$$  \hspace{1cm} (2.14)

where $C(t, \varepsilon) = \{ \tau \in C : |\tau - t| < \varepsilon \}$ and the weight $r(t)$ is defined by (2.2), is bounded in the space $L^p(C)$ with $p(t)$ satisfying assumptions (1.3) and (1.4) on $C$, if and only if conditions (2.4) are fulfilled.

3 Statements of the main results

For the weighted singular operator

$$S_\rho f(x) = \rho(x) \int_{\mathbb{R}^1} \frac{f(y) \, dy}{\rho(y)(y-x)}, \quad x \in \mathbb{R}^1,$$  \hspace{1cm} (3.1)

where $\rho(x) = |x - x_0|^\mu(1 + |x|)^\nu$, we prove the following theorem.

**Theorem A.** Let $p(x)$ satisfy assumptions (1.3)-(4.12) and (2.1). The operator $S_\rho$ is bounded in the space $L^p(\mathbb{R}^1)$ if and only if

$$-\frac{1}{p(x_0)} < \mu < \frac{1}{q(x_0)}, \quad -\frac{1}{p(\infty)} < \mu + \nu < \frac{1}{q(\infty)}.$$  \hspace{1cm} (3.2)

We also consider the weighted potential type operator of variable order

$$I_{\beta, \gamma, \mu, \nu}^{\alpha(\cdot)} f(x) = |x - x_0|^\beta(1 + |x|)^\gamma \int_{\mathbb{R}^1} \frac{f(y) \, dy}{|y - x_0|^\mu(1 + |y|)^\nu |x - y|^{1 - \alpha(x)}}, \quad x_0 \in \mathbb{R}^1$$  \hspace{1cm} (3.3)

and prove the following statement.

**Theorem B.** Let $p(x)$ satisfy assumptions (1.3)-(1.4) and (2.1) and $\alpha(x)$ satisfy assumptions (2.8) and (2.9) on $\Omega = \mathbb{R}^1$ and condition (2.1). Then
1) the operator $I_{\beta,\gamma,\mu,\nu}^{\alpha}$ is bounded in the space $L^{p(\cdot)}(\mathbb{R}^1)$ if

$$\beta = \mu - \alpha(x_0), \quad \gamma = \nu - \alpha(\infty), \quad \left(\alpha(\infty) = \lim_{|x| \to \infty} \alpha(x)\right)$$

(3.4)

and

$$\alpha(x_0) - \frac{1}{p(x_0)} < \mu < \frac{1}{q(x_0)}, \quad \alpha(\infty) - \frac{1}{p(\infty)} < \nu < \frac{1}{q(\infty)};$$

(3.5)

2) In the case $\min_{x \in \mathbb{R}^1} \alpha(x)p(x) < 1$ the operator $I_{\beta,\gamma,\mu,\nu}^{\alpha}$ with

$$\beta = \mu = 0 \quad \text{and} \quad \gamma = 1 - \frac{2}{p(\infty)} - \alpha(\infty), \quad \nu = 1 - \frac{2}{p(\infty)} + \alpha(\infty)$$

(3.6)

is also bounded from the space $L^{p(\cdot)}(\mathbb{R}^1)$ into $L^{p_{\alpha}(\cdot)}(\mathbb{R}^1)$, where $\frac{1}{p_{\alpha}(x)} = \frac{1}{p(x)} - \alpha(x)$.

Observe that a result similar to statement 2) of Theorem B in the case of the half-axis $\mathbb{R}^1_+$ and special values of weight exponents was earlier obtained for fractional integrals in [3], see Theorems 2.3 and 2.4 in [3].

4 Proof of Theorems A and B.

The proof may be obtained via the mapping of $\mathbb{R}^1$ onto the unit circle. For simplicity we take $x_0 = 0$.

Let $\Gamma = \{t \in \mathbb{C} : |t| = 1\}$. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$ we put

$$\frac{x - i}{x + i} = t, \quad \frac{y - i}{y + i} = \tau \in \Gamma$$

(4.1)

so that

$$x = i\frac{1 + t}{1 - t}, \quad y = i\frac{1 + \tau}{1 - \tau} \in \mathbb{R}^1$$

for $t, \tau \in \Gamma$.

In the sequel we use the notation

$$p^*(t) = p\left(i\frac{1 + t}{1 - t}\right), \quad \alpha^*(t) = \alpha\left(i\frac{1 + t}{1 - t}\right), \quad t \in \Gamma.$$
Lemma 4.1. The function $p(x)$ satisfies both conditions (1.4) and (2.1) on $\mathbb{R}^1$, if and only if the function $p^*(t)$ satisfies the condition

$$|p^*(t) - p^*(\tau)| \leq \frac{B}{\ln \frac{3}{|t - \tau|}}$$

for all $t, \tau \in \Gamma$ with some $B > 0$.

Proof. It is easily seen that both conditions (1.4) and (2.1) may be written in a unified way:

$$|p(x) - p(y)| \leq \frac{A}{\ln^{\sqrt{1+x^2}} \sqrt{1+y^2}}$$

which is nothing else but (4.2).

Let $K(x, y)$ be any kernel and

$$A_{p}f(x) = \rho_1(x) \int_{\mathbb{R}^1} \frac{K(x, y)}{\rho_2(y)} f(y) \, dy$$

(4.3)

where

$$\rho_1(x) = |x|^\beta (1 + |x|)^\gamma, \quad \rho_2(x) = |x|^\mu (1 + |x|)^\nu.$$

Under passage (4.1) we have

$$\rho_1 \left( \frac{i + t}{1 - t} \right) \approx |1 + t|^{\beta} |1 - t|^{-\beta - \gamma}, \quad \rho_2 \left( \frac{i + t}{1 - t} \right) \approx |1 + t|^{\mu} |1 - t|^{-\mu - \nu}$$

so that

$$\left| A_{p}f \left( \frac{i + t}{1 - t} \right) \right| \approx \frac{|1 + t|^{\beta}}{|1 - t|^{\beta + \gamma}} \int_{\Gamma} \frac{K \left( \frac{i + t}{1 - t}, \frac{i + \tau}{1 - \tau} \right)}{|1 + \tau|^\mu |1 - \tau|^\frac{2}{q^*(i)} - \mu - \nu} \alpha(\tau)\psi(\tau) \, d\tau$$

(4.4)

where $\frac{1}{q^*(i)} = 1 - \frac{1}{p^*(i)}$, $\alpha(\tau) = \frac{|1 - \tau|^2}{(1 - \tau)^2} \left( |1 - \tau| + |1 + \tau| \right)^\nu$, and

$$\psi(\tau) = \frac{f \left( \frac{i + \tau}{1 - \tau} \right)}{|1 - \tau|^\frac{2}{p^*(i)}}.$$

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It is easily seen that
\[
\int_{\mathbb{R}^3} |f(x)|^p(x) \, dx = 2 \int_{\Gamma} \frac{|f \left( i \frac{1+t}{1-t} \right) |^{p(t)}}{|1-t|^2} \, |dt| \approx \int_{\Gamma} \frac{|\psi(t)|^{p(t)}}{|1-t|^2} \, |dt| \quad (4.5)
\]
where we have used the fact that
\[
|1-t|^{\frac{2}{p(t)}} \approx |1-t|^{\frac{2}{p(t)}}
\]
(4.6)
in view of condition (4.2). Therefore,
\[
\| f \|_{L^p(\mathbb{R}^1)} \approx \| \psi \|_{L^p(\mathbb{R}^1)}, \quad (4.7)
\]
Note that \( a(t) \) is a function bounded from below and above: \(|a(\tau)| = (|1-\tau| + |1+\tau|)^\nu \), so that
\[
2^\nu \leq |a(\tau)| \leq (2\sqrt{2})^\nu, \quad \text{if} \quad \nu \geq 0
\]
and
\[
(2\sqrt{2})^\nu \leq |a(\tau)| \leq 2^\nu, \quad \text{if} \quad \nu \leq 0
\]
which is easily checked for \( \tau = e^{i\theta} \).

**A. The case of the singular operator.** We choose \( K(x,y) = \frac{1}{x-y} \)
and \( \beta = \mu \) and \( \gamma = \nu \) in (4.3) so that the operator \( A_\rho \) turns to be the singular operator \( S_\rho \) defined in (3.3). Then
\[
\mathcal{K} \left( i \frac{1+t}{1-t}, i \frac{1+\tau}{1-\tau} \right) = 2i \frac{(1-t)(1-\tau)}{\tau - t}
\]
and relation (4.4) takes the form
\[
\left| S_\rho f \left( i \frac{1+t}{1-t} \right) \right| \approx \frac{|1+t|^\mu}{|1-t|^\mu+\nu} \left| \int_{\Gamma} \frac{|1-\tau|^\mu+\nu+1-\sigma^2(t)}{|1+\tau|^\mu} \tilde{f}(\tau) \, d\tau \right| \quad (4.8)
\]
where
\[
\tilde{f}(\tau) = a(\tau) \psi(\tau) = a(\tau) \cdot \frac{f \left( i \frac{1+\tau}{1-\tau} \right)}{|1-\tau|^2} \quad (4.9)
\]
and
\[ \int_{\mathbb{R}^1} |f(x)|^{p(x)} \, dx \approx \int_{\Gamma} |\tilde{f}(t)|^{p^*(t)} \, |dt|. \] 
(4.10)

From (4.8) according to (4.10)-(4.9) we have
\[ \int_{\mathbb{R}^1} |S_\rho f(x)|^{p(x)} \, dx \approx \int_{\Gamma} \left| \frac{1}{t^{\frac{2}{p^*(t)}}} (S_\rho f)(i \frac{1 + t}{1 - t}) \right|^{p^*(t)} \, |dt| \]
\[ \approx \int_{\Gamma} \left| r(t) \int_{\Gamma} \frac{\tilde{f}(\tau) \, d\tau}{r(\tau)(\tau - t)} \right|^{p^*(t)} \, |dt| \] 
(4.11)

where
\[ r(t) = |1 + t|^\mu |1 - t|^\nu, \quad \nu_1 = 1 - \mu - \nu - \frac{2}{p^*(1)}. \]

Taking Lemma 4.1 into account, we apply Theorem 2.1 and conclude that the operator $S_\rho$ is bounded in the space $L^{p(x)}(\mathbb{R}^1)$ if and only if
\[ -\frac{1}{p^*(-1)} < \mu < \frac{1}{q^*(-1)} \quad \text{and} \quad -\frac{1}{p^*(1)} < \nu_1 < \frac{1}{q^*(1)}. \] 
(4.12)

Since $p^*(-1) = p(0)$ and $p^*(1) = p(\infty)$, it is easy to check that conditions (4.12) coincide with assumptions (3.2), which proves Theorem A.

**B). The case of the potential operator.** Under the choice $K(x, y) = |x - y|^{\alpha(x)}^{-1}$ we have
\[ K \left( i \frac{1 + t}{1 - t}, i \frac{1 + \tau}{1 - \tau} \right) \approx \frac{|1 - t|^{1 - \alpha^*(1)} |1 - \tau|^{1 - \alpha^*(1)}}{|\tau - t|^{1 - \alpha^*(t)}} \]
and then from (4.4) we have
\[ \left| \left( I_{\beta, \gamma, \mu, \nu} f \right)(i \frac{1 + t}{1 - t}) \right| \approx \frac{|1 + t|^\beta}{|1 - t|^{\beta + \gamma + \alpha^*(1) - 1}} \int_{\Gamma} \frac{|1 - t|^{\mu + \nu + 1 - \frac{2}{p^*(1)} - \alpha^*(1)} \psi(\tau) \, d\tau}{|1 + t|^\mu |\tau - t|^{1 - \alpha^*(t)}}. \] 
(4.13)

1) The $L^{p(\cdot)} \rightarrow L^{p(\cdot)}$ estimate.
From (4.13) according to (4.5) and (4.6) we have

$$\int_{\mathbb{R}^1} \left| \left( I_{\alpha, \gamma, \mu, \nu}^{(\cdot)} \right) (x) \right|^{p(x)} dx \approx \int_{\Gamma} \left| 1 + t \right|^{\beta} \left| 1 - t \right|^{\gamma_1} \int_{\Gamma} \frac{\psi(\tau)}{ \left| 1 + \tau \mu \right| \left| 1 - \tau \nu \right| \left| \tau - t \right|^{1 - \alpha^* (t)} } dx dt,$$

where

$$\gamma_1 = 1 - \frac{2}{p^*(1)} - \beta - \gamma - \alpha^*(1), \quad \gamma_2 = 1 - \frac{2}{p^*(1)} - \mu - \nu + \alpha^*(1).$$

By Corollary 2.5, we arrive at the boundedness statement under the conditions

$$\beta = \mu - \alpha^*(-1), \quad \gamma = \nu - \alpha^*(1)$$

and

$$\alpha^*(-1) - \frac{1}{p^*(-1)} < \mu < \frac{1}{q^*(-1)}, \quad \alpha^*(1) - \frac{1}{p^*(1)} < \mu + \nu < \frac{1}{q^*(1)},$$

which coincide with conditions (3.4)-(3.5) of Theorem B. Part 1 of Theorem B is proved.

2) The \( L^{p(\cdot)} \rightarrow L^{p_0(\cdot)} \) estimate.

Similarly, from (4.13) according to (4.5) and (4.6) in the case (3.6) we have

$$\int_{\mathbb{R}^1} \left| \left( I_{\alpha, \gamma, \mu, \nu}^{(\cdot)} \right) (x) \right|^{p_0(\cdot)} dx$$

$$\approx \int_{\Gamma} \left| 1 - t \right|^{\gamma + \alpha^*(1) - 1 + \frac{2}{p^*(1)}} \int_{\Gamma} \frac{1 - \tau^{1 + \nu - \alpha^*(1) - \frac{2}{q^*(1)}}} \left| \tau - t \right|^{1 - \alpha^*(t)} \psi(\tau) d\tau $$

$$\gamma + \alpha^*(1) - 1 + \frac{2}{p^*(1)} = 0 \quad \text{and} \quad 1 + \nu - \alpha^*(1) - \frac{2}{q^*(1)} = 0$$

since

by (3.6).

Making use of Corollary 2.3, we arrive at statement 2) of Theorem B.
5 On the boundedness of the maximal operator on $\mathbb{R}^1$.

Let

$$M^\rho f(x) = \operatorname*{sup}_{h>0} M_h^\rho f(x), \quad \text{where} \quad M_h^\rho f(x) = \frac{\rho(x)}{2h} \int_{x-h}^{x+h} \frac{|f(y)|}{\rho(y)} \, dy \quad (5.1)$$

where $\rho(x) = |x - x_0|^\mu (1 + |x|)^\nu$.

**Hypothesis.** Under conditions (1.1)-(1.2) and (2.1)

$$\| M^\rho f(x) \|_{p(\cdot)} \leq C \| f \|_{p(\cdot)} \quad (5.2)$$

with $C > 0$ not depending on $f$, if

$$-\frac{1}{p(x_0)} < \mu < \frac{1}{q(x_0)}, \quad -\frac{1}{p(\infty)} < \mu + \nu < \frac{1}{q(\infty)}. \quad (5.3)$$

We are able to prove this statement in a weaker form, namely for the maximal operator defined by (5.1), but with $\operatorname*{sup}_{0<h<1} M_h^\rho f(x)$ instead of $\operatorname*{sup}_{h>0} M_h^\rho f(x)$. Namely, the following theorem is valid.

**Theorem C.** Under assumptions (1.3)-(1.4) and (2.1)

$$\left\| \operatorname*{sup}_{0<h<1} M_h^\rho f(x) \right\|_{p(\cdot)} \leq C \| f \|_{p(\cdot)} \quad (5.4)$$

if conditions (5.3) are satisfied.

Proof. As in the proof of Theorem A, we map $\mathbb{R}^1$ onto the unit circle $\Gamma$, but this time this procedure with respect to the maximal operator is more complicated. Under change of variables (4.1) we have

$$(M_h^\rho f)(x)|_{x=i\frac{1+t}{1-t}} \approx \frac{\rho^*(t)}{2h} \int_{\phi_1(h)}^{\phi_2(h)} \frac{|f(\tau)|}{\rho^*(\tau)} \frac{|d\tau|}{|1 - \tau|^2} \quad (5.5)$$

where

$$\rho^*(t) = |1 + t|^\mu |1 - t|^{-\mu - \nu}$$
and
\[ \varphi_1(h) = \varphi_1(h,t) = \arg \frac{x - h - i}{x - h + i}, \quad \varphi_2(h) = \varphi_2(h,t) = \arg \frac{x + h - i}{x + h + i}. \]

Since \( \arg(x - i) = \pi - \arctg \frac{1}{x} \) for \( x < 0 \) and \( \arg(x - i) = -\arctg \frac{1}{x} \) for \( x > 0 \), we have
\[ \varphi := \arg \frac{x - i}{x + i} = 2 \arg(x - i) = -2 \arctg \frac{1}{x} \]
and
\[ \varphi_1(h) = -2 \arctg \frac{1}{x - h}, \quad \varphi_2(h) = -2 \arctg \frac{1}{x + h} \]

In (5.5) we integrate along the arc \( \varphi_1(h) \leq \theta \leq \varphi_2(h), \ \theta = \arg \tau \), which is not symmetric with respect to its ”center” \( t = e^{i\varphi}, \ \varphi = -2 \arctg \frac{1}{x} \):
\[ \varphi_2(h) - \varphi \neq \varphi - \varphi_1(h) \]
(but \( \lim_{h \to 0} \frac{\varphi_2(h) - \varphi}{\varphi - \varphi_1(h)} = 1 \)). Indeed, we have
\[ \varphi_2(h) - \varphi = 2 \left[ \arctg \frac{1}{x} - \arctg \frac{1}{x + h} \right], \ \varphi - \varphi_1(h) = 2 \left[ \arctg \frac{1}{x - h} - \arctg \frac{1}{x} \right] \]
so that
\[ \varphi_2(h) - \varphi = 2 \arctg \frac{h}{x^2 + hx + 1} = 2 \arctg \frac{h}{(x + \frac{h}{2})^2 + 1 - \frac{h^2}{4}}, \]
\[ \varphi - \varphi_1(h) = 2 \arctg \frac{h}{x^2 - hx + 1} = 2 \arctg \frac{h}{(x - \frac{h}{2})^2 + 1 - \frac{h^2}{4}}. \]

Therefore, for \( h \leq 2 \) we have
\[ \varphi_2(h) - \varphi \leq \varphi - \varphi_1(h), \ \text{if} \quad x > 0, \]
\[ \varphi_2(h) - \varphi \geq \varphi - \varphi_1(h), \ \text{if} \quad x < 0. \]

Let
\[ \delta = \delta(h,x) = \max \{ \varphi_2(h) - \varphi, \varphi - \varphi_1(h) \} = 2 \arctg \frac{h}{x^2 - h|x| + 1}. \quad (5.6) \]

We introduce the function
\[ \psi(t) = \frac{f \left( \frac{t^{1+t}}{1-t} \right)}{|1-t|^\frac{2}{p(t)}} \in L^p(t)(\Gamma) \]
where $p^*(t) = p(\frac{1+t}{1-t})$ is the same as in the proof of Theorems A and B.

From (5.5) we have

$$\int \left| (M^p_h f)(x) \right|^{p(x)} dx \leq \int \left| \sup_{0 < \theta < 1} \frac{\rho_2(t)}{2h} \int_{\varphi - \delta}^{\varphi + \delta} \left| \psi(t) \right| \rho_1(\tau) d\tau \right|^{p^*(t)} dt, \quad (5.7)$$

where

$$\rho_1(t) = p^*(t) \cdot |1 - t|^{\frac{2}{\varphi(\delta)}} = |1 + t|^\mu |1 - t|^{-\mu - \nu - \frac{\delta}{\varphi(\delta)}}$$

and

$$\rho_2(t) = \rho_1(t) \cdot |1 - t|^{-2} = |1 + t|^\mu |1 - t|^{-\mu - \nu - \frac{\delta}{\varphi(\delta)}}.$$ 

It remains to pass from $h$ to $\delta$ in (5.7).

From (5.6) we observe that

$$\frac{1}{h} = \frac{1 + |x|tg \frac{\delta}{2}}{(1 + |x|^2)tg \frac{\delta}{2}}. \quad (5.8)$$

According to (5.6) we also see that

$$|x|tg \frac{\delta}{2} = \frac{|x|h}{x^2 - h|x| + 1} \leq 1 \quad \text{for} \quad 0 < h \leq 1.$$ 

Therefore,

$$\frac{1}{h} \leq \frac{2}{(1 + |x|^2)tg \frac{\delta}{2}} \leq \frac{c |t - 1|^2}{tg \frac{\delta}{2}} \leq \frac{c |t - 1|^2}{\delta}. \quad (5.9)$$

Since $h \leq 1$, we have $|y - x| \leq 1$ so that

$$\frac{1}{2}(1 + |x|) \leq (1 + |y|) \leq 2(1 + |x|)$$

from which it follows that

$$c_1 |1 - t| \leq |1 - \tau| \leq c_2 |1 - t|.$$ 

Then from (5.9) we have

$$\frac{1}{h} \leq c \frac{|t - 1| \cdot |\tau - 1|}{\delta}.$$
Consequently, from (5.7)

\[ \int_{\mathbb{R}^1} |(M^p_h f)(x)|^{p(x)} \, dx \leq c \int \left( \sup_{0 < h < 1} \frac{r(t)}{2\delta} \int_{\varphi - \delta}^{\varphi + \delta} \frac{\psi(t)}{r(\tau)} \, d\tau \right)^{p^*(t)} \, dt, \quad (5.10) \]

where \( r(t) = |t + 1| - 1|^{-\mu - \nu + 1 - \frac{2}{p^*(t)}} \) and

\[ \delta = \delta(h, t) = 2 \arctg \frac{h}{x^2 - h|x| + 1}. \]

Then from (5.10)

\[ \int_{\mathbb{R}^1} |(M^p_h f)(x)|^{p(x)} \, dx \leq \int \left( \sup_{0 < \delta < \pi} \frac{r(t)}{2\delta} \int_{\varphi - \delta}^{\varphi + \delta} \frac{\psi(t)}{r(\tau)} \, d\tau \right)^{p^*(t)} \, dt. \quad (5.11) \]

By Corollary 2.7, we arrive at the boundedness statement if the conditions

\[ -\frac{1}{p^*(-1)} < \mu < \frac{1}{q^*(-1)}, \quad -\frac{1}{p^*(1)} < -\mu - \nu + \frac{2}{q^*(1)} < \frac{1}{q^*(-1)} \]

are satisfied, which coincide with assumptions (5.3).

\[ \square \]

References


