The classical integral operators in weighted Lorentz spaces with variable exponent

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Abstract

In this paper the Lorentz spaces with variable exponent are introduced. These Banach function spaces are defined on the base of variable Lebesgue spaces. Boundedness of classical integral operators are proved in variable Lorentz spaces.

Introduction

The Lebesgue and Sobolev spaces with variable exponent have been studied intensively by many mathematicians (see e. g. the survey paper [1]). The study of those spaces has been stimulated by various problems of calculus of variation and differential equations with nonstandard growth conditions, theory of elasticity, fluid mechanics.

In this paper we study the mapping properties of classical operators arising in Harmonic Analysis, PDE, and Boundary value problems for analytic functions in new Banach function spaces with variable exponent.

1. On Some Banach Function Spaces

Let p be a measurable function on $(0,a)\,, 0<a\leq\infty$ such that

$$1<\underline{p}=\underset{(0,a)}{ess\ inf}\ p(t)\leq p(t)\leq \underset{(0,a)}{ess\ sup}\ p(t)=\overline{p}<\infty.$$

By $L^p(0,a)$ we denote the space of measurable functions $g(t)$ on $(0,a)$ such that

$$I_p(g)=\int\limits_0^a |g(t)|^{p(t)}\,dt<\infty.$$

$L^{p(\cdot)}$ is a Banach function space with respect to the norm

$$\|g\|_{L^{p(\cdot)}_{(0,a)}}=\inf_{\lambda>0}\left\{\lambda>0:I_p\left(\frac{g}{\lambda}\right)\leq 1\right\}.$$

Let Ω be a bounded domain in R^n and let $f:\Omega\longrightarrow R^1$ be a measurable function, where Ω is a bounded domain in R^n. By

$$f^*(t)=\sup\{s\geq 0:m(x\in\Omega:|f(x)|>s)>t\}$$

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we denote the nonincreasing rearrangement of a function f, where m is the Lebesgue measure. It is clear that $f^*(t) = 0$ for $t > m\Omega$, since $m\Omega < \infty$.

Variable Lorentz spaces $\Lambda^{p(\cdot)}(\Omega)$ is introduced as a set of all measurable functions for which

$$\|f^*\|_{L^{p(\cdot)}(0,m\Omega)} < \infty.$$

The last is a quasinorm. The norm in $\Lambda^{p(\cdot)}$ is defined throught the average

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(y)\,dy, \qquad f^*(t) \le f^{**}(t).$$

Definition 1. *The subset of all measurable functions on* Ω *for which*

$$\|f\|_{\Lambda^{p(\cdot)}} = \|f^{**}\|_{L^{p(\cdot)}(0,m\Omega)} < \infty$$

is called the space $\Lambda^{p(\cdot)}(\Omega)$.

When the function $p(x)$ is such that in $L^{p(\cdot)}$ Hardy's inequality holds, we conclude that

$$\|f^*\|_{L^{p(\cdot)}} \le \|f\|_{\Lambda^{p(\cdot)}} \le c\, \|f^*\|_{L^{p(\cdot)}}.$$

Note that $\|f^{**}\|_{L^{p(\cdot)}}$ is a norm. The triangle inequality follows from the inequality

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t),$$

(see e. g. [2]).

In [3] it is proved that $\Lambda^{p(\cdot)}$ is a Banach space.

2. Integral Operators in $\Lambda^{p(\cdot)}$

In this section we present some results on boundedness of maximal functions, singular integral and Riesz potential in $\Lambda^{p(\cdot)}$.

In the sequel the main tool of our investigation will Hardy's inequality in $L^{p(\cdot)}$.

Theorem A [4] *Let*

$$f \to \mathcal{P}f$$

be the Hardy's operator

$$\mathcal{P}f(x) = \frac{1}{x}\int_0^x f(y)\,dy.$$

Let $I = [0,a]$ *for* $a < \infty$ *and let* $p : I \to [1,\infty)$ *be bounded,* $p(0) > 1$ *and*

$$\lim_{x\to 0+} (p(x) - p(0)) \log\frac{1}{x} < \infty$$

and

$$\overline{p}_{(0,x_0)} = \underset{(0,x_0)}{ess\ sup}\, p(x) = p(0)$$

for some $x_0 \in (0, a)$. *If* $\beta \in \left[0, 1 - \frac{1}{p(0)}\right)$, *then Hardy's inequality*

$$\left\| \mathcal{P}f(x)\, x^{\beta} \right\|_{L^{p(\cdot)}} \leq c \left\| f(x)\, x^{\beta} \right\|_{L^{p(\cdot)}}$$

holds.

We are interested in that function $p(x)$ for which the conjugate to $\mathcal{P}$ is also bounded in $L^{p'(\cdot)}$.

It should be noted that if p continuous at $t = 0$, then with the conditions mentined in Theorem A the function $p'(x)$ satisfied the same conditions.

Consequently from the duality argument when $\beta = 0$ we conclude that if p is continuous at zero, then under the conditions of Theorem A

$$Qf(x) = \int_x^a \frac{f(y)}{y} dy$$

is bounded in $L^{p(\cdot)}$ as well.

Definition 2. 1. *The function* p *said to be of class* $\mathcal{D}$ *if it is continuous at* $t = 0$ *and satisfies the conditions of Theorem A.*

Theorem 2. 1. *Let* p *be as in Theorem A. Then the Hardy-Littlewood function*

$$Mf(x) = \sup_{Q} \frac{1}{m(Q \cap \Omega)} \int_{Q \cap \Omega} |f(y)|\, dy$$

is bounded in $\Lambda^{p(\cdot)}_{x^{\beta}}$, *i. e.*

$$\left\| (Mf)^{**} x^{\beta} \right\|_{L^{p(\cdot)}} \leq c \left\| f^{**} x^{\beta} \right\|_{L^{p(\cdot)}},$$

when $0 \leq \beta < 1 - \frac{1}{p(0)}$.

Proof. The validness of this statement follows from the relation (see, e. g. [2] and [5])

$$(Mf)^{*}(f) \sim \frac{1}{t} \int_0^t f^{*}(y)\, dy$$

and Theorem A.

Now we consider the mapping property of Calderon-Zygmund singular integrals

$$Kf(x) = V.P. \int_{\mathbb{R}^n} \frac{K(y)}{|y|^n} f(x - y)\, dy, \quad x \in \Omega$$

where K is an odd function on $\mathbb{R}^n$ homogeneus of degree 0 and satisfying the Dini condition on the unit sphere S^{n-1}

$$\int_0^2 \frac{\omega(\delta)}{\delta} d\delta < \infty,$$

where

$$\omega(\delta) = \sup_{\substack{x,y \in S^{n-1} \\ |x-y|<\delta}} |K(x) - K(y)|.$$

As a particular case one may mention the finite Hilbert transform $\left(n = 1, K(x) = \frac{x}{|x|}\right)$ and the Riesz transform $\left(n \geq 2, K(x) = \frac{x_j}{|x|}, j = 1, 2, ..., n\right)$.

Theorem 2. 2. *Let $p \in \mathcal{D}$. Then the operator K is bounded in $\Lambda^{p(\cdot)}(\Omega)$.*

Proof. Thanks to the known estimate we have that

$$(Kf)^*(t) \leq c\left(\frac{1}{t}\int_0^t f^*(u)\,du + \int_t^{m\Omega} \frac{f^*(u)}{u} du\right).$$

Now it is sufficient to apply Theorem A and the remark to this statement.

Theorem 2. 3. *Let $p \in \mathcal{D}$. Then the Riesz potential*

$$(I_\alpha f)(x) = \int_G \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

over the bounded domain acts boundedly from $\Lambda^{p(\cdot)}$ into itself.

Proof. We use the estimate (see [2])

$$(I_\alpha f)^*(t) \leq c\left(t^{\alpha-1}\int_0^t f^*(u)\,du + \int_t^{m\Omega} f^*(u)\,u^{-1+\alpha} du\right).$$

Then it is easy to see that

$$(I_\alpha f)^*(t) \leq c\,(m\Omega)^\alpha \left(\frac{1}{t}\int_0^t f^*(u)\,du + \int_t^{m\Omega} \frac{f^*(u)}{u} du\right).$$

The proof we complete as in previous theorem.

3. On Cauchy Singular Integrals in Lorentz Spaces with Variable Exponent

In this section we consider the Cauchy singular integral

$$(S_\Gamma f)(t) = \frac{1}{\pi i}\int\limits_\Gamma \frac{f(\tau)}{\tau - t}d\tau, \quad t = t(s), 0 \le s \le l,$$

where Γ is a finite rectifiable Jordan curve on which the arc length is chosen as a parameter, starting from any fixed point.

Γ is called Lyapunov curve if $t'(s) \in Lip\alpha, 0 < \alpha \le 1$. When $t'(s)$ is a function of bounded variation, Γ is called a curve of bounded turning.

The aim is to study the mapping properties of S_Γ in Lorentz spaces with variable exponent.

We assume that $p(s)$ is defined on $[0, l]$, where l is the length of Γ. In previous sections we introduced the class of p functions: $p \in \mathcal{D}$, if the functions p and p' satisfy the conditions of Theorem A.

By $L^{p(\cdot)}(\Gamma)$ we denote the space of all measurable functions $f(t(s)) = f_0(s)$ on $[0, l]$, for which

$$I_p(f_0) = \int\limits_0^l |f_0(s)|^{p(s)} ds < \infty.$$

$L^{p(\cdot)}(\Gamma)$ is a Banach function space with respect to the norm

$$\|f\|_{L^{p(\cdot)}} = \inf_\lambda \left\{\lambda > 0 : I_p\left(\frac{f_0}{\lambda}\right) \le 1\right\}.$$

By $\Lambda^{p(\cdot)}(\Gamma)$ we denote the space of all measurable functions on Γ for which

$$\|f^*\|_{L^{p(\cdot)}} < \infty,$$

where f^* denotes the non-increasing rearrangement of a function f.

$\Lambda^{p(\cdot)}(\Gamma)$ is a Banach function space by the norm

$$\|f\|_{\Lambda^{p(\cdot)}} = \|f^{**}\|_{L^{p(\cdot)}},$$

where as above

$$f^{**}(t) = \frac{1}{t}\int\limits_0^t f^*(u)\, du.$$

Since $p \in \mathcal{D}$, the Hardy's inequality holds, thus

$$\|f\|_{\Lambda^{p(\cdot)}} \sim \|f^*\|_{L^{p(\cdot)}}.$$

It means that to state boundedness of some integral operator T in $\Lambda^{p(\cdot)}$ it is sufficient to estimate of quasinorms $\|(Tf)^*\|_{L^{p(\cdot)}}$.

Theorem 3. 1. *Let $p \in \mathcal{D}$ and let Γ is a Lyapunov curve Then S_Γ is bounded in $\Lambda^{p(\cdot)}$.*

Proof. In the case of Lyapunov curve the estimate

$$|(S_\Gamma f)(t)| \le \left(\int_0^l \frac{f_0(s)}{\sigma - s} ds + \int_0^l \frac{|f_0(s)|}{|s-\sigma|^{1-\alpha}} ds \right), \quad t = t(\sigma)$$

holds with $\alpha \in (0,1)$ (see, e. g. [6]).

Hence

$$((S_\Gamma f))^*(t) \le c\left((Hf_0)^* + (I_\alpha f_0)^*\right),$$

where H is the finite Hilbert transform and I_α is the fractional integral of order α.

Now applying the Theorems 2. 2 and 2. 3 we obtain the desired result.

Theorem 3. 2. *Let Γ be a curve of bounded turning without cusps. Let $p \in \mathcal{D}$. Then the operator S_Γ is bounded in $\Lambda^{p(\cdot)}$.*

Proof. It is obvious that

$$(S_\Gamma f)(t) = \int_0^l \frac{f_0(s)}{\sigma - s} d\sigma + \int_0^l \left(\frac{t'(\sigma)}{t(\sigma) - t(s)} - \frac{1}{\sigma - s} \right) f_0(\sigma)\, d\sigma, \quad t = t(s),$$

as $t'(s)$ is a function of bounded variation, we have

$$\left| t'(s) - t'(\sigma) \right| \le |V(s) - V(\sigma)|,$$

where $V(s)$ is the total variation of t' on $[0,s]$. Let

$$Tf(t) = \int_0^l h(\sigma, s) f_0(\sigma)\, d\sigma, \quad t = t(s),$$

where

$$h(\sigma, s) = \frac{t'(\sigma)}{t(\sigma) - t(s)} - \frac{1}{\sigma - s}.$$

Since Γ is a curve of bounded turning without cusps we have

$$0 < c_1 < \left| \frac{t(s) - t(\sigma)}{s - \sigma} \right|.$$

Therefore we can derive the estimate (see, e. g. [3])

$$\begin{aligned}|h(\sigma,s)| &\leq \left|\frac{t'(\sigma)}{t(\sigma)-t(s)}-\frac{1}{\sigma-s}\right| \\ &\leq \frac{c}{(s-\sigma)^2}\left|\int_s^\sigma\left[t'(s)-t'(u)\right]du\right| \leq \frac{c}{(s-\sigma)^2}\left|\int_\sigma^s\left[V(s)-V(u)\right]du\right| \\ &\leq c(s-\sigma)^{-1}(V(s)-V(\sigma)).\end{aligned}$$

Thus

$$\begin{aligned}|Tf(t)| &\leq c\int_0^l \frac{V(s)-V(\sigma)}{s-\sigma}f_0(\sigma)\,d\sigma \\ &\leq c\left(V(s)\left|\int_0^l \frac{f_0(\sigma)}{s-\sigma}d\sigma\right| + \left|\int_0^l \frac{V(\sigma)f_0(\sigma)}{s-\sigma}d\sigma\right|\right).\end{aligned}$$

Now since V is bounded, passing to the nonincreasing rearrangement in the last inequality, in virtue of Theorem 2. 2 we conclude the boundedness of S_Γ in $\Lambda^{p(\cdot)}$.

Note that for the constant function $p(s)=p$ the boundedness of S_Γ on Lyapunov curve and on curve of bounded turning without cusps was proved in [6] and [7], respectively.

It should be noted that the paper [3] is pioneering in which it has been introduced weighted variable Lorentz spaces and proved the boundedness of classical integral operators but under more restrictive condition for the function $p(s)$.

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