On the mean summability by Cesaro \((C, a)\) and Abel-Poisson methods of trigonometric Fourier series in the weighted Lorentz spaces

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Abstract

In the paper the classical results on the mean convergence and summability of trigonometric Fourier series are extended to the weighted Lorentz spaces. 2000 Mathematics Subject Classification: 46E30.

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Let \(T\) be the interval \((-\pi, \pi)\). In the theory of trigonometric Fourier series it is well known (see, [1]) that Cesaro and Abel-Poisson means converges in \(L^p(T)\) \((1 \leq p \leq \infty)\). The problem of mean summability in weighted Lebesgue spaces has been investigated in [2], [3] in the frame of \(A_p\) classes.

In the present paper we study the mean summability problems in weighted Lorentz spaces. Let \(f\) be \(2\pi\)-periodic measurable function. Then let \(w\) be \(2\pi\)-periodic nonnegative integrable on \(T\). The last functions are called as weights. For the Borel set \(E\) by

\[
we = \int_E w(x) \, dx
\]

we denote the Borel measure generated by the function \(w\). For the function \(f\) consider its non-increasing rearrangement with respect to measure \(w\):

\[
f^*(t) = \sup \{ s \geq 0 : w(\{ x \in T : |f(x)| > s \} > t) \}
\]

Then consider the average of \(f^*\):

\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(y) \, dy.
\]

It is easy to see that \(f^*(t) = 0\), when \(t > 2\pi\). Let \(1 < p, s < \infty\). The weighted Lorentz space \(L_w^{p,s}(T)\) is defined as the set of all measurable functions \(f\), for which

\[
\|f\|_{L_w^{p,s}} = \left( \int_0^{2\pi} (f^{**}(t))^{ts} \frac{dt}{t} \right)^{1/s} < \infty.
\]

It is known, that \(L_w^{p,s}(T)\) is a Banach space (see, example e. g. [4]).

Let \(f \in L^1(T)\) and

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

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be the Fourier series of function $f \in L^1(T)$. Let

$$
\sigma_n^\alpha(x,f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) K_n^\alpha(t) dt, \quad \alpha > 0
$$

(5)

when

$$
K_n^\alpha(t) = \sum_{k=0}^{n} \frac{A_n^{-k} D_k(t)}{A_n^\alpha},
$$

with

$$
D_k(t) = \frac{\sin \left( \frac{k+1}{2} t \right)}{2 \sin \left( \frac{1}{2} t \right)}
$$

and

$$
A_n^\alpha = \left( \frac{n + \alpha}{n} \right) \approx \frac{n^\alpha}{\Gamma(\alpha + 1)}.
$$

Definition. A weight function $w$ is said to be of class $A_p$ if

$$
\sup_{I} \left| I \right| \int_I w(x) dx \left( \frac{1}{\left| I \right|} \int_I w^{-\frac{1}{p}}(x) dx \right)^{p-1} < \infty,
$$

where the least upper bound is taken over all intervals $I$, the length of which are not greater than $2\pi$.

Let

$$
\tilde{f}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{ix}) cotg \frac{1}{2} x dt
$$

and

$$
Mf(x) = \sup_{h > 0} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)| dt.
$$

In the sequel we need the following two statements.

Proposition 1. Let $1 < p, s < \infty$. Then the operator $\tilde{f}$ is bounded in $L^p_w$ if and only if $w \in A_p$.

Proposition 2. Let $1 < p, s < \infty$. Then the operator $M$ is bounded in $L^p_w$ if and only if $w \in A_p$.

For the proof of these Propositions see [4] and [5].

Let

$$
\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)
$$

be the conjugate series to the Fourier series (4). Let $S_n(x,f)$ and $\tilde{S}_n(x,f)$ denote the partial sums of (4) and (6) respectively.

Theorem 1. Let $1 < p < \infty$ and $w \in A_p$. Then we have

$$
i) \lim_{n \to \infty} \|S_n(\cdot,f) - f\|_{L^p_w} = 0
$$
and
\[ ii) \lim_{n \to \infty} \left\| \sigma_n(\cdot, f) - f \right\|_{L^p_w} = 0. \]

The proof of Theorem is similar as in classical case for \( L^p \) spaces, basing on Proposition 1 (see, [1]).

**Theorem 2.** Let \( 1 < p, s < \infty \) and let \( w \in A_p \). Then
\[ \lim_{n \to \infty} \left\| \sigma_n(\cdot, f) - f \right\|_{L^p_w} = 0, \quad 0 < \alpha < 1. \]

For \( 0 \leq r < 1 \) let
\[ U_r(x, f) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n \]
be the Abel-Poisson means of function \( f \).

**Theorem 3.** Let \( 1 < p, s < \infty \) and let \( w \in A_p \). Then
\[ \lim_{r \to 1} \left\| U_r(\cdot, f) - f \right\|_{L^p_w} = 0. \]

Note that from Proposition 2 and the following estimates (see, [1])
\[ \sup_{0 < r < 1} |U_r(x, f)| \leq cM f(x) \]
and for positive functions \( f \)
\[ c_1 M f(x) \leq \sup_{0 < r < 1} |U_r(x, f)| \]
we conclude

**Proposition 3.** Let \( 1 < p, s < \infty \). Then the operator
\[ Nf(x) = \sup_{0 < r < 1} |U_r(x, f)| \]
is bounded in \( L^p_w \) if and only if \( w \in A_p \).

**Proposition 4.** Let \( 1 < p, s < \infty \) and let \( w \in A_p \). Then the operator
\[ f \to \sup_{n \geq 1} |\sigma_n(\cdot, f)| \]
is bounded in \( L^p_w \).

The last Proposition follows from the known estimate (see [1])
\[ \sup_{n \geq 1} |\sigma_n(x, f)| \leq cM f(x) \]
and the Proposition 2.

**The proof of Theorem 2.** Let us consider the sequence of operator
\[ U_n : f \to \sigma_n(\cdot, f), \quad n = 1, 2, .... \]
Let us show that each $U_n$ is linear and bounded in $L^p_w$. The linearity is clear. Applying the estimate

$$K_n^\alpha(t) < 2n$$

we get

$$\|\sigma_n^\alpha(\cdot, f)\|_{L^p_w} \leq c_\alpha \left\| \int_{-\pi}^{\pi} |f(t)| dt \right\|_{L^p_w}.$$ 

But using the generalized Holder’s inequality for $L^p_w$ (see e.g. [5]) we get

$$\int |f(t)| dt \leq \|f\|_{L^p_w} \cdot \|l\|_{L^{p'}_w} \leq c \|f\|_{L^p_w}.$$ 

Thus the operator $U_n$ is bounded for each $n$. On the other hand by Proposition 4, the sequence of operator norms

$$\|U_n\|_{L^p_w \rightarrow L^p_w}$$

is bounded.

The set of continuous functions is dense in $L^p_w$ and the Fourier series of continuous functions on the real line converges uniformly to $f$. Thus the Cesaro means of continuous functions converges in the norm $L^p_w$. Applying the Banach-Steinhaus theorem we conclude that we have the $(C, \alpha)$ summability for arbitrary function $f \in L^p_w$.

Theorem is proved.

References


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